ROME Example: Portfolio Selection

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1 Introduction

In this example, we demonstrate how ROME can be used to model and solve a portfolio selection problem. We briefly review the problem here and refer interested readers to Goh and Sim [2] for a more detailed discussion of the model. Given a collection of risky assets, we aim to construct a portfolio of these assets which minimizes some risk metric. The decisions are the weights of the respective assets in the collection. We use as our objective the Conditional Value-at-Risk (CVaR) risk metric (popularized by Rockafellar and Uryasev [3]).

We are given historical returns on the assets. In this example, we demonstrate how to use ROME to solve the portfolio selection problem using three methods, namely (1) by sampling, (2) by using moments, and (3) by using segregated moments.

2 Model Description

Notation: We let N denote the total number of assets which available for investment. Our decision is a vector $\boldsymbol{x} \in \Re^N$, with the i^{th} component representing the fraction of our net wealth to be invested in asset i. The drivers of uncertainty in this model are the asset returns, which we denote by the uncertainty vector $\tilde{\boldsymbol{r}} \in \Re^N$. We do not assume the precise distribution of $\tilde{\boldsymbol{r}} \in \Re^N$, but instead, we assume that we have accurate estimates of the asset return means $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$, which characterizes a family of distributions $\mathbb{F}$. We aim to find a portfolio allocation which minimizes its CVaR, subject to attaining an exogenously specified mean target return τ.

2.1 General Model

Adapting the definition of CVaR in [3] to our robust scenario, we use CVaR_β as our objective:

$$\text{CVaR}_\beta(\boldsymbol{x}) \triangleq \min_{v\in\Re} \left\{ v + \frac{1}{1-\beta} \sup_{\mathbb{P}\in\mathbb{F}} \text{E}_{\mathbb{P}} \left(\left(-\tilde{\boldsymbol{r}}'\boldsymbol{x} - v \right)^+ \right) \right\}. \tag{2.1}$$

Putting this together with the standard portfolio constraints, we have the model:

$$
\begin{aligned}
Z_{PORT} = \quad \min_{\boldsymbol{x},v} \quad & v + \frac{1}{1-\beta} \sup_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}} \left(\left(-\tilde{\boldsymbol{r}}'\boldsymbol{x} - v \right)^{+} \right) \\
\text{s.t.} \quad & \boldsymbol{\mu}'\boldsymbol{x} \geq \tau \\
& \boldsymbol{e}'\boldsymbol{x} = 1 \\
& \boldsymbol{x} \geq \boldsymbol{0}.
\end{aligned}
\tag{2.2}
$$

We compare three methods of solving problem the portfolio optimization problem (2.2). In all of the methods which we study, we presume to have asset returns data structured into two groups: an in-sample period, and an out-of-sample period. We take the in-sample data as returns data that has already been revealed to the modeler, and hence can be used in the modeling and portfolio construction. Conversely, we take the out-of-sample data as model uncertainties, unknown to the modeler at the point of constructing the portfolio.

2.2 Sampling Approach

We begin with a sampling approach as a benchmark, where we use historical samples to approximate the expectation term in the objective of problem (2.2). Denoting the number of trading days in the in-sample period as T, and the realized in-sample returns as $\left\{\boldsymbol{r}^t\right\}_{t=1}^{T}$, the explicit model can be written as

$$
\begin{aligned}
Z_{PORT}{}^{(1)} = \quad \min_{\boldsymbol{x},v} \quad & v + \frac{1}{(1-\beta)T} \sum_{t=1}^{T} \left(-\boldsymbol{r}^{t\prime}\boldsymbol{x} - v \right)^{+} \\
\text{s.t.} \quad & \boldsymbol{\mu}'\boldsymbol{x} \geq \tau \\
& \boldsymbol{e}'\boldsymbol{x} = 1 \\
& \boldsymbol{x} \geq \boldsymbol{0}.
\end{aligned}
\tag{2.3}
$$

2.3 Using Moments

Our second approach is to linearize problem (2.2). To this end, we introduce a scalar-valued Linear Decision Rule (LDR) auxilliary variable $y(\tilde{\boldsymbol{r}})$ to linearize the objective and fit this into the distributionally robust optimization framework of Goh and Sim [1]. We obtain the transformed model:

$$
\begin{aligned}
Z_{PORT}{}^{(2)} = \quad \min_{\boldsymbol{x},v} \quad & v + \frac{1}{1-\beta} \sup_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}} \left(y(\tilde{\boldsymbol{r}}) \right) \\
\text{s.t.} \quad & \boldsymbol{\mu}'\boldsymbol{x} \geq \tau \\
& \boldsymbol{e}'\boldsymbol{x} = 1 \\
& \boldsymbol{x} \geq \boldsymbol{0} \\
& y(\tilde{\boldsymbol{r}}) \geq -\tilde{\boldsymbol{r}}'\boldsymbol{x} - v \\
& y(\tilde{\boldsymbol{r}}) \geq 0 \\
& y(\tilde{\boldsymbol{r}}) \text{ is an LDR.}
\end{aligned}
\tag{2.4}
$$

The family of distributions $\mathbb{F}$ is defined by the mean and covariance of the uncertainties, estimated from the sample mean and covariance during the in-sample period, i.e.

$$\mathbb{F} = \left\{ \mathbb{P} : \mathrm{E}_{\mathbb{P}}\left(\tilde{\boldsymbol{r}}\right) = \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{r}^t, \mathrm{E}_{\mathbb{P}}\left(\tilde{\boldsymbol{r}}\tilde{\boldsymbol{r}}'\right) = \hat{\boldsymbol{\Gamma}} : \hat{\Gamma}_{ij} = \frac{1}{T-1}\sum_{t=1}^{T}\left(r_i^t r_j^t\right) \right\}. \tag{2.5}$$

2.4 Using Segregated Moments

Our final method is in all respects identical to the second method, with the sole exception that we partition the uncertainties into positive and negative half-spaces, and use the segregated uncertainties and moments instead. Denoting the segregated uncertainties as $\tilde{\boldsymbol{s}}$, this can be explicitly written as:

$$\tilde{\boldsymbol{s}} = \begin{bmatrix} \tilde{\boldsymbol{r}}^+ \\ \tilde{\boldsymbol{r}}^- \end{bmatrix}. \tag{2.6}$$

Consequently, we can reconstitute $\tilde{\boldsymbol{r}}$ from $\tilde{\boldsymbol{s}}$ as $\tilde{\boldsymbol{r}} = \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{I} \end{bmatrix} \tilde{\boldsymbol{s}}$. Using segregated uncertainties, Problem (2.2) can be explicitly written as:

$$\begin{array}{rll} Z_{PORT}{}^{(3)} = & \min\limits_{\boldsymbol{x},v} & v + \dfrac{1}{1-\beta}\sup\limits_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(y(\tilde{\boldsymbol{s}})\right) \\ & \text{s.t.} & \boldsymbol{\mu}'\boldsymbol{x} \geq \tau \\ & & \boldsymbol{e}'\boldsymbol{x} = 1 \\ & & \boldsymbol{x} \geq \boldsymbol{0} \\ & & y(\tilde{\boldsymbol{s}}) \geq -\left(\begin{bmatrix} \boldsymbol{I} & -\boldsymbol{I} \end{bmatrix}\tilde{\boldsymbol{s}}\right)'\boldsymbol{x} - v \\ & & y(\tilde{\boldsymbol{s}}) \geq 0 \\ & & y(\tilde{\boldsymbol{s}}) \text{ is an LDR.} \end{array} \tag{2.7}$$

with the corresponding family of uncertainties:

$$\mathbb{F} = \left\{ \mathbb{P} : \mathbb{P}(\tilde{\boldsymbol{s}} \geq \boldsymbol{0}) = 1, \mathrm{E}_{\mathbb{P}}\left(\tilde{\boldsymbol{s}}\right) = \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{s}^t, \mathrm{E}_{\mathbb{P}}\left(\tilde{\boldsymbol{s}}\tilde{\boldsymbol{s}}'\right) = \hat{\boldsymbol{\Gamma}} : \hat{\Gamma}_{ij} = \frac{1}{T-1}\sum_{t=1}^{T}\left(s_i^t s_j^t\right) \right\}, \tag{2.8}$$

where the derived in-sample data $\left\{\boldsymbol{s}^t\right\}_{t=1}^{T}$ is given by

$$\boldsymbol{s}^t = \begin{bmatrix} \left(\boldsymbol{r}^t\right)^+ \\ \left(\boldsymbol{r}^t\right)^- \end{bmatrix} \quad \forall t \in [T]. \tag{2.9}$$

References

[1] J. Goh and M. Sim. Distributionally robust optimization and its tractable approximations. *Working Paper*, 2009.

[2] J. Goh and M. Sim. Robust Optimization Made Easy with ROME. *Working Paper*, 2009.

[3] R. T. Rockafellar and S. Uryasev. Optimization of conditional value-at-risk. *Journal of Risk*, 2:493–517, 2000.