Online appendix of "Distributionally Robust Optimization and its Tractable Approximations"

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Appendix A Proofs of LDR and SLDR reductions

A.1 Proof of Proposition 1

We use $\mathcal{L}(m_k, N, I_k)$ to approximate $\mathcal{Y}(m_k, N, I_k)$ in Problem (2.2). Applying the definition of $\mathcal{L}(m_k, N, I_k)$ in (4.1) for each $k \in [K]$, the problem above equivalently becomes:

$$
\begin{array}{lll}
\min\limits_{\boldsymbol{x}, \left\{\boldsymbol{y}^{0,k}, \boldsymbol{Y}^k\right\}_{k=1}^K} & \boldsymbol{c}^{0\prime}\boldsymbol{x} + \sup\limits_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\sum\limits_{k=1}^K \boldsymbol{d}^{0,k\prime}\left(\boldsymbol{y}^{0,k} + \boldsymbol{Y}^k\tilde{\boldsymbol{z}}\right)\right) & \\
\text{s.t.} & \boldsymbol{c}^{l\prime}\boldsymbol{x} + \sup\limits_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\sum\limits_{k=1}^K \boldsymbol{d}^{l,k\prime}\left(\boldsymbol{y}^{0,k} + \boldsymbol{Y}^k\tilde{\boldsymbol{z}}\right)\right) \le b_l & \forall l \in [M] \\
& \boldsymbol{T}(\tilde{\boldsymbol{z}})\boldsymbol{x} + \sum\limits_{k=1}^K \boldsymbol{U}^k\left(\boldsymbol{y}^{0,k} + \boldsymbol{Y}^k\tilde{\boldsymbol{z}}\right) = \boldsymbol{v}(\tilde{\boldsymbol{z}}) & \\
& \underline{\boldsymbol{y}}^k \le \boldsymbol{y}^{0,k} + \boldsymbol{Y}^k\tilde{\boldsymbol{z}} \le \overline{\boldsymbol{y}}^k & \forall k \in [K] \\
& \boldsymbol{Y}^k\boldsymbol{e}^j = \boldsymbol{0} & \forall j \notin I_k, \forall k \in [K] \\
& \boldsymbol{x} \ge \boldsymbol{0}. &
\end{array}
\tag{A.1}
$$

We now proceed to show that Problems (A.1) and (4.2) are equivalent. We first notice that due to linearity, the expectation terms in the objective first M constraints can be expressed as:

$$
\begin{aligned}
\sup_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\sum_{k=1}^K \boldsymbol{d}^{l,k\prime}\left(\boldsymbol{y}^{0,k} + \boldsymbol{Y}^k\tilde{\boldsymbol{z}}\right)\right) &= \sum_{k=1}^K \boldsymbol{d}^{l,k\prime}\boldsymbol{y}^{0,k} + \sup_{\mathbb{P}\in\mathbb{F}}\left(\sum_{k=1}^K \boldsymbol{d}^{l,k\prime}\boldsymbol{Y}^k \mathrm{E}_{\mathbb{P}}\left(\tilde{\boldsymbol{z}}\right)\right) \\
&= \sum_{k=1}^K \boldsymbol{d}^{l,k\prime}\boldsymbol{y}^{0,k} + \sup_{\hat{\boldsymbol{z}}\in\hat{\mathcal{W}}}\left(\sum_{k=1}^K \boldsymbol{d}^{l,k\prime}\boldsymbol{Y}^k\hat{\boldsymbol{z}}\right),
\end{aligned}
$$

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for any $l \in \{0\} \cup [M]$. Next, for the following constraints to hold for the random variable $\tilde{\boldsymbol{z}}$,

$$\begin{aligned}
\boldsymbol{T}(\tilde{\boldsymbol{z}})\boldsymbol{x} + \sum_{k=1}^{K} \boldsymbol{U}^k \left(\boldsymbol{y}^{0,k} + \boldsymbol{Y}^k \tilde{\boldsymbol{z}}\right) &= \boldsymbol{v}(\tilde{\boldsymbol{z}}) \\
\underline{\boldsymbol{y}}^k \leq \boldsymbol{y}^{0,k} + \boldsymbol{Y}^k \tilde{\boldsymbol{z}} \leq \overline{\boldsymbol{y}}^k & \qquad \forall k \in [K],
\end{aligned}$$

it is necessary and sufficient for the constraints to hold within the support, i.e.

$$\begin{aligned}
\boldsymbol{T}(\boldsymbol{z})\boldsymbol{x} + \sum_{k=1}^{K} \boldsymbol{U}^k \left(\boldsymbol{y}^{0,k} + \boldsymbol{Y}^k \boldsymbol{z}\right) &= \boldsymbol{v}(\boldsymbol{z}) \quad \forall \boldsymbol{z} \in \mathcal{W} \\
\underline{\boldsymbol{y}}^k \leq \boldsymbol{y}^{0,k} + \boldsymbol{Y}^k \boldsymbol{z} \leq \overline{\boldsymbol{y}}^k & \qquad \forall \boldsymbol{z} \in \mathcal{W}, \forall k \in [K].
\end{aligned}$$

Since the model data $\boldsymbol{T}(\cdot), \boldsymbol{v}(\cdot)$ are assumed to be affine in their respective arguments, we can equivalently re-write the equality constraint as a sum of the components of $\boldsymbol{z}$, as:

$$\left(\boldsymbol{T}^0 \boldsymbol{x} + \sum_{k=1}^{K} \boldsymbol{U}^k \boldsymbol{y}^{0,k} - \boldsymbol{v}^0\right) + \sum_{i=1}^{N} z_i \left(\boldsymbol{T}^i \boldsymbol{x} + \sum_{k=1}^{K} \boldsymbol{U}^k \boldsymbol{Y}^k \boldsymbol{e}^i - \boldsymbol{v}^i\right) = \boldsymbol{0} \quad \forall \boldsymbol{z} \in \mathcal{W}.$$

Finally, since $\mathcal{W}$ is assumed to be full-dimensional, the constraint holds iff the individual coefficients vanish, i.e.

$$\begin{aligned}
\boldsymbol{T}^0 \boldsymbol{x} + \sum_{k=1}^{K} \boldsymbol{U}^k \boldsymbol{y}^{0,k} &= \boldsymbol{v}^0 \\
\boldsymbol{T}^i \boldsymbol{x} + \sum_{k=1}^{K} \boldsymbol{U}^k \boldsymbol{Y}^k \boldsymbol{e}^i &= \boldsymbol{v}^i \quad \forall i \in [N].
\end{aligned}$$

Putting these all together, Problems (A.1) and (4.2) are equivalent as desired. ■

A.2 Proof of Proposition 2

For an arbitrary $\boldsymbol{z} \in \Re^N$, we consider $\boldsymbol{\zeta} = \boldsymbol{M}(\boldsymbol{z})$, and the components of $\boldsymbol{\gamma} = \boldsymbol{F}\boldsymbol{\zeta}$, $\forall i \in [N]$,

$$\begin{aligned}
\gamma_i &= \boldsymbol{e}^{i\prime} \boldsymbol{F} \boldsymbol{\zeta} \\
&= \sum_{l=0}^{L-1} \boldsymbol{e}^{i+lN\prime} \boldsymbol{\zeta} \quad \text{(by structure of } \boldsymbol{F}\text{)} \\
&= \sum_{j \in \Phi(i)} \zeta_j.
\end{aligned} \tag{A.2}$$

Where the set $\Phi(i)$ is defined for brevity as

$$\Phi(i) \triangleq \{j \in [N_E] : i = ((j-1) \bmod N) + 1\}. \tag{A.3}$$

Firstly, for each $i \in [N]$, we first consider the case where $\exists j \in \Phi(i), \zeta_j \notin \{\xi_{i,k}\}_{k=2}^{L}$. We can omit $k = 1$ and $k = L + 1$ in the consideration of the set above, since $\zeta_j \neq \pm\infty$. In this case, since $\{\xi_{i,k}\}_{k=1}^{L+1}$ segments the extended real line, $\exists k^* \in [L]$ such that $\xi_{i,k^*} < \zeta_{j^*} < \xi_{i,k^*+1}$ for some $j^* \in \Phi(i)$.

Furthermore, using the definition of a segregation (4.6), $\zeta_{j^*} = z_i$, and $\xi_{i,k^*} < z_i < \xi_{i,k^*+1}$. Using property (4.5), $\forall k \in [L+1]$, we get

$$\begin{array}{lcll} z_i & > & \xi_{i,k} & \text{if} \quad k < k^*, \\ z_i & < & \xi_{i,k} & \text{if} \quad k > k^*. \end{array}$$

Hence, using the definition of a segregation (4.6) again, and recalling that $k = \lceil j/N \rceil$, this implies that $\forall j \in \Phi(i)$,

$$\begin{array}{lcll} \zeta_j & = & \xi_{i,k+1} & \text{if} \quad j < j^*, \\ \zeta_j & = & \xi_{i,k} & \text{if} \quad j > j^*, \\ \zeta_j & = & z_i & \text{if} \quad j = j^*. \end{array}$$

Hence, substituting into (A.2), we get

$$\begin{aligned} \gamma_i &= \sum_{j \in \Phi(i)} \zeta_j \\ &= \zeta_{j^*} + \sum_{\substack{j \in \Phi(i) \\ j < j^*}} \zeta_j + \sum_{\substack{j \in \Phi(i) \\ j > j^*}} \zeta_j \\ &= z_i + \sum_{k=2}^{L} \xi_{i,k} \\ &= z_i - g_i. \end{aligned}$$

Since this holds for each $i \in [N]$, we obtain $\boldsymbol{FM}(\boldsymbol{z}) = \boldsymbol{z} - \boldsymbol{g}$, $\forall \boldsymbol{z} \in \Re^N$ such that $\exists j \in \Phi(i), \boldsymbol{e}^{j\prime}\boldsymbol{M}(\boldsymbol{z}) \notin \{\xi_{i,k}\}_{k=2}^{L}$.

Next, we consider the case that $\forall j \in \Phi(i), \zeta_j \in \{\xi_{i,k}\}_{k=2}^{L}$. There are L elements in $\Phi(i)$, and ζ_j can take on $L-1$ distinct values. Hence, we can apply the pigeonhole principle, which implies that $\exists k^* \in \{2, \ldots, L\}, \exists j_1, j_2 \in \Phi(i)$, such that $\zeta_{j_1} = \zeta_{j_2} = \xi_{i,k^*}$. From the definition of a segregation, (4.6), we can establish that $|j_1 - j_2| = N$, and $\zeta_{j_1} = \zeta_{j_2} = z_i$. We can express this alternatively as $\exists j^* \in \Phi(i)$, such that $z_i = \zeta_{j^*} = \zeta_{j^*+N} = \xi_{i,k^*}$. Finally, recalling that $k = \lceil j/N \rceil$, we again have

$$\begin{array}{lcll} \zeta_j & = & \xi_{i,k+1} & \text{if} \quad j < j^*, \\ \zeta_j & = & \xi_{i,k} & \text{if} \quad j > j^*, \\ \zeta_j & = & \xi_{i,k^*} = z_i & \text{if} \quad j = j^*. \end{array}$$

By the same argument as the previous case, we obtain $\boldsymbol{z} = \boldsymbol{FM}(\boldsymbol{z}) + \boldsymbol{g}$, $\forall \boldsymbol{z} \in \Re^N$ such that $\forall j \in \Phi(i), \boldsymbol{e}^{j\prime}\boldsymbol{M}(\boldsymbol{z}) \in \{\xi_{i,k}\}_{k=2}^{L}$. Combining these two cases allows us to conclude that $\forall \boldsymbol{z} \in \Re^N$, $\boldsymbol{z}$=$\boldsymbol{FM}(\boldsymbol{z})$+$\boldsymbol{g}$. ■

A.3 Proof of Proposition 3

To prove the first inequality, we express $\boldsymbol{\zeta} = \boldsymbol{M}(\boldsymbol{z})$, and equivalently re-express Problem (4.7) as

$$\begin{array}{llll}
{Z_{SLDR}}^* = & \displaystyle\min_{\boldsymbol{x},\{\boldsymbol{r}^k(\cdot)\}_{k=1}^K} & \boldsymbol{c}^{0\prime}\boldsymbol{x} + \sum_{k=1}^K \boldsymbol{d}^{0,k\prime}\boldsymbol{r}^{0,k} + \sup_{\hat{\boldsymbol{\zeta}}\in\hat{\mathcal{V}}^*}\left(\sum_{k=1}^K \boldsymbol{d}^{0,k\prime}\boldsymbol{R}^k\hat{\boldsymbol{\zeta}}\right) & \\
& \text{s.t.} & \boldsymbol{c}^{l\prime}\boldsymbol{x} + \sum_{k=1}^K \boldsymbol{d}^{l,k\prime}\boldsymbol{r}^{0,k} + \sup_{\hat{\boldsymbol{\zeta}}\in\hat{\mathcal{V}}^*}\left(\sum_{k=1}^K \boldsymbol{d}^{l,k\prime}\boldsymbol{R}^k\hat{\boldsymbol{\zeta}}\right) \le b_l & \forall l \in [M] \\
& & \boldsymbol{T}(\boldsymbol{F}\boldsymbol{\zeta}+\boldsymbol{g})\boldsymbol{x} + \left(\sum_{k=1}^K \boldsymbol{U}^k\boldsymbol{r}^{0,k} + \boldsymbol{U}^k\boldsymbol{R}^k\boldsymbol{\zeta}\right) = \boldsymbol{v}(\boldsymbol{F}\boldsymbol{\zeta}+\boldsymbol{g}) \quad \forall \boldsymbol{\zeta}\in\mathcal{V}^* & \\
& & \underline{\boldsymbol{y}}^k \le \boldsymbol{r}^{0,k} + \boldsymbol{R}^k\boldsymbol{\zeta} \le \overline{\boldsymbol{y}}^k \quad \forall \boldsymbol{\zeta}\in\mathcal{V}^* & \forall k\in[K] \\
& & \boldsymbol{x} \ge \boldsymbol{0} & \\
& & \boldsymbol{r}^k \circ \boldsymbol{M} \in \mathcal{Y}(m_k, N, I_k) & \forall k \in [K]\,.
\end{array} \tag{A.4}$$

We assume that we have some $\left(\boldsymbol{x}, \left\{\boldsymbol{r}^{0,k}, \boldsymbol{R}^k\right\}_{k=1}^K\right)$ that is feasible in the approximated SLDR problem (4.12). The inclusion $\hat{\mathcal{V}}^* \subseteq \hat{\mathcal{V}}$, implies that the first M inequalities in Problem (A.4) are satisfied. Furthermore, the inclusion $\mathcal{V}^* \subseteq \mathcal{V}$ implies that the upper and lower bounds in Problem (A.4) are also satisfied. To show that the equality constraint in Problem (A.4) is satisfied, we consider the following expression for an arbitrary $\boldsymbol{\zeta} \in \mathcal{V}^*$:

$$\begin{aligned}
&\boldsymbol{T}(\boldsymbol{F}\boldsymbol{\zeta}+\boldsymbol{g})\boldsymbol{x} + \left(\sum_{k=1}^K \boldsymbol{U}^k\boldsymbol{r}^{0,k} + \boldsymbol{U}^k\boldsymbol{R}^k\boldsymbol{\zeta}\right) - \boldsymbol{v}(\boldsymbol{F}\boldsymbol{\zeta}+\boldsymbol{g}) \\
=\;& \left(\boldsymbol{T}^0\boldsymbol{x} + \sum_{i=1}^N (\boldsymbol{e}^{i\prime}\boldsymbol{F}\boldsymbol{\zeta} + g_i)\boldsymbol{T}^i\boldsymbol{x}\right) + \left(\sum_{k=1}^K \boldsymbol{U}^k\boldsymbol{r}^{0,k} + \boldsymbol{U}^k\boldsymbol{R}^k\boldsymbol{\zeta}\right) - \left(\boldsymbol{v}^0 + \sum_{i=1}^N (\boldsymbol{e}^{i\prime}\boldsymbol{F}\boldsymbol{\zeta} + g_i)\boldsymbol{v}^i\right) \\
=\;& \left(\boldsymbol{T}^0\boldsymbol{x} + \sum_{i=1}^N g_i\boldsymbol{T}^i\boldsymbol{x} + \sum_{k=1}^K \boldsymbol{U}^k\boldsymbol{r}^{0,k} - \boldsymbol{v}^0 - \sum_{i=1}^N g_i\boldsymbol{v}^i\right) + \sum_{i=1}^N (\boldsymbol{e}^{i\prime}\boldsymbol{F}\boldsymbol{\zeta})\left(\boldsymbol{T}^i\boldsymbol{x} - \boldsymbol{v}^i\right) + \sum_{k=1}^K \boldsymbol{U}^k\boldsymbol{R}^k\boldsymbol{\zeta} \\
=\;& \left(\boldsymbol{\mathcal{T}}^0\boldsymbol{x} + \sum_{k=1}^K \boldsymbol{U}^k\boldsymbol{r}^{0,k} - \boldsymbol{\nu}^0\right) + \sum_{i=1}^N (\boldsymbol{e}^{i\prime}\boldsymbol{F}\boldsymbol{\zeta})\left(\boldsymbol{T}^i\boldsymbol{x} - \boldsymbol{v}^i\right) + \sum_{k=1}^K \boldsymbol{U}^k\boldsymbol{R}^k\boldsymbol{\zeta} \\
=\;& \left(\boldsymbol{\mathcal{T}}^0\boldsymbol{x} + \sum_{k=1}^K \boldsymbol{U}^k\boldsymbol{r}^{0,k} - \boldsymbol{\nu}^0\right) + \sum_{i=1}^N \sum_{j=1}^{N_E} F_{ij}\zeta_j\left(\boldsymbol{T}^i\boldsymbol{x} - \boldsymbol{v}^i\right) + \sum_{k=1}^K \boldsymbol{U}^k\boldsymbol{R}^k \sum_{i=j}^{N_E} \zeta_j\boldsymbol{e}^j \\
=\;& \left(\boldsymbol{\mathcal{T}}^0\boldsymbol{x} + \sum_{k=1}^K \boldsymbol{U}^k\boldsymbol{r}^{0,k} - \boldsymbol{\nu}^0\right) + \sum_{j=1}^{N_E} \zeta_j \sum_{i=1}^N F_{ij}\left(\boldsymbol{T}^i\boldsymbol{x} - \boldsymbol{v}^i\right) + \sum_{j=1}^{N_E} \zeta_j \sum_{k=1}^K \boldsymbol{U}^k\boldsymbol{R}^k\boldsymbol{e}^j \\
=\;& \left(\boldsymbol{\mathcal{T}}^0\boldsymbol{x} + \sum_{k=1}^K \boldsymbol{U}^k\boldsymbol{r}^{0,k} - \boldsymbol{\nu}^0\right) + \sum_{j=1}^{N_E} \zeta_j\left(\boldsymbol{\mathcal{T}}^j\boldsymbol{x} - \boldsymbol{\nu}^j + \sum_{k=1}^K \boldsymbol{U}^k\boldsymbol{R}^k\boldsymbol{e}^j\right).
\end{aligned}$$

Hence, the system of equality constraints in Problem (4.12) implies that the above expression vanishes component-wise for any $\boldsymbol{\zeta} \in \Re^{N_E}$, which in turn implies that it vanishes component-wise for any $\boldsymbol{\zeta} \in \mathcal{V}^*$. Finally, we consider the non-anticipativity requirement. Denoting by $(\boldsymbol{M}(\boldsymbol{z}))_j$ the j^{th} component of

$\boldsymbol{M}(\boldsymbol{z})$ for some $j \in [N_E]$,

$$j \in \Phi_k \Leftrightarrow \left(\boldsymbol{M}\left(\boldsymbol{z} + \sum_{i \notin I_k} \lambda_i \boldsymbol{e}^i\right)\right)_j = \left(\boldsymbol{M}\left(\boldsymbol{z}\right)\right)_j \quad \forall \boldsymbol{\lambda} \in \Re^N \tag{A.5}$$

for each $k \in [K]$. The forward direction follows directly from the definition of Φ_k in (4.11). The reverse direction results from requiring the equality to hold for all $\boldsymbol{\lambda} \in \Re^N$. The only components $(\boldsymbol{M}(\boldsymbol{z}))_j$ that are invariant to all $\boldsymbol{\lambda}$ are those with indices j in the set $\{j \in [N_E] : (i-1) \equiv (j-1) \bmod N\} = \Phi_k$.

Next, we expand the composite function:

$$\begin{aligned}
\boldsymbol{r}^k \circ \boldsymbol{M}\left(\boldsymbol{z}\right) &= \boldsymbol{r}^{0,k} + \boldsymbol{R}^k \boldsymbol{M}\left(\boldsymbol{z}\right) \\
&= \boldsymbol{r}^{0,k} + \boldsymbol{R}^k \sum_{j=1}^{N_E} \left(\boldsymbol{M}\left(\boldsymbol{z}\right)\right)_j \boldsymbol{e}^j \\
&= \boldsymbol{r}^{0,k} + \sum_{j=1}^{N_E} \left(\boldsymbol{M}\left(\boldsymbol{z}\right)\right)_j \boldsymbol{R}^k \boldsymbol{e}^j \\
&= \boldsymbol{r}^{0,k} + \sum_{j \in \Phi_k} \left(\boldsymbol{M}\left(\boldsymbol{z}\right)\right)_j \boldsymbol{R}^k \boldsymbol{e}^j,
\end{aligned} \tag{A.6}$$

where the last equality is due to the assumption of feasibility in (4.12), which gives $\boldsymbol{R}^k \boldsymbol{e}^j = \boldsymbol{0}, \forall j \notin \Phi_k$. Hence, for an arbitrary $\boldsymbol{\lambda} \in \Re^N$,

$$\begin{aligned}
\boldsymbol{r}^k \circ \boldsymbol{M}\left(\boldsymbol{z} + \sum_{i \notin I_k} \lambda_i \boldsymbol{e}^i\right) &= \boldsymbol{r}^{0,k} + \sum_{j \in \Phi_k} \left(\boldsymbol{M}\left(\boldsymbol{z} + \sum_{i \notin I_k} \lambda_i \boldsymbol{e}^i\right)\right)_j \boldsymbol{R}^k \boldsymbol{e}^j \quad \text{(by (A.6))} \\
&= \boldsymbol{r}^{0,k} + \sum_{j \in \Phi_k} \left(\boldsymbol{M}\left(\boldsymbol{z}\right)\right)_j \boldsymbol{R}^k \boldsymbol{e}^j \quad \text{(by (A.5))} \\
&= \boldsymbol{r}^k \circ \boldsymbol{M}\left(\boldsymbol{z}\right) \quad \text{(by (A.6))},
\end{aligned}$$

implying that $\boldsymbol{r}^k \circ \boldsymbol{M} \in \mathcal{Y}(m_k, N, I_k)$ as required. Therefore, we have established that any feasible solution to (4.12) is always feasible in (A.4), with an objective that is not smaller. Hence we have ${Z_{SLDR}}^* \leq Z_{SLDR}$.

To prove the second inequality, we consider Problem (4.12), and choose $\forall k \in [K]$,

$$\begin{aligned}
\boldsymbol{R}^k &= \boldsymbol{Y}^k \boldsymbol{F}, \\
\boldsymbol{r}^{0,k} &= \boldsymbol{y}^{0,k} + \boldsymbol{Y}^k \boldsymbol{g}.
\end{aligned}$$

Problem (4.12) becomes:

$$
\begin{aligned}
Z_{SLDR} = \quad & \min_{\boldsymbol{x},\left\{\boldsymbol{y}^{0,k},\boldsymbol{Y}^k\right\}_{k=1}^K} && \boldsymbol{c}^{0\prime}\boldsymbol{x} + \sum_{k=1}^K \boldsymbol{d}^{0,k\prime}\boldsymbol{y}^{0,k} + \sup_{\hat{\boldsymbol{\zeta}}\in\hat{\mathcal{V}}}\left(\sum_{k=1}^K \boldsymbol{d}^{0,k\prime}\boldsymbol{Y}^k(\boldsymbol{F}\hat{\boldsymbol{\zeta}}+\boldsymbol{g})\right) \\
& \text{s.t.} && \boldsymbol{c}^{l\prime}\boldsymbol{x} + \sum_{k=1}^K \boldsymbol{d}^{l,k\prime}\boldsymbol{y}^{0,k} + \sup_{\hat{\boldsymbol{\zeta}}\in\hat{\mathcal{V}}}\left(\sum_{k=1}^K \boldsymbol{d}^{l,k\prime}\boldsymbol{Y}^k(\boldsymbol{F}\hat{\boldsymbol{\zeta}}+\boldsymbol{g})\right) \leq b_l \quad \forall l \in [M] \\
& && \boldsymbol{T}^0\boldsymbol{x} + \left(\sum_{k=1}^K \boldsymbol{U}^k\boldsymbol{y}^{0,k} + \boldsymbol{U}^k\boldsymbol{Y}^k\boldsymbol{g}\right) = \boldsymbol{\nu}^0 \\
& && \boldsymbol{T}^j\boldsymbol{x} + \sum_{k=1}^K \boldsymbol{U}^k\boldsymbol{Y}^k\boldsymbol{F}\boldsymbol{e}^j = \boldsymbol{\nu}^j \qquad \forall j \in [N_E] \\
& && \underline{\boldsymbol{y}}^k \leq \boldsymbol{y}^{0,k} + \boldsymbol{Y}^k(\boldsymbol{F}\boldsymbol{\zeta}+\boldsymbol{g}) \leq \overline{\boldsymbol{y}}^k \quad \forall \boldsymbol{\zeta}\in\mathcal{V} \qquad \forall k \in [K] \\
& && \boldsymbol{Y}^k\boldsymbol{F}\boldsymbol{e}^j = \boldsymbol{0} \qquad \forall j \notin \Phi_k, \forall k \in [K] \\
& && \boldsymbol{x} \geq \boldsymbol{0}.
\end{aligned}
$$

Expanding the terms $\left\{\boldsymbol{T}^j, \boldsymbol{\nu}^j\right\}_{j=0}^{N_E}$, we obtain

$$
\begin{aligned}
Z_{SLDR} = \quad & \min_{\boldsymbol{x},\left\{\boldsymbol{y}^{0,k},\boldsymbol{Y}^k\right\}_{k=1}^K} && \boldsymbol{c}^{0\prime}\boldsymbol{x} + \sum_{k=1}^K \boldsymbol{d}^{0,k\prime}\boldsymbol{y}^{0,k} + \sup_{\hat{\boldsymbol{\zeta}}\in\hat{\mathcal{V}}}\left(\sum_{k=1}^K \boldsymbol{d}^{0,k\prime}\boldsymbol{Y}^k(\boldsymbol{F}\hat{\boldsymbol{\zeta}}+\boldsymbol{g})\right) \\
& \text{s.t.} && \boldsymbol{c}^{l\prime}\boldsymbol{x} + \sum_{k=1}^K \boldsymbol{d}^{l,k\prime}\boldsymbol{y}^{0,k} + \sup_{\hat{\boldsymbol{\zeta}}\in\hat{\mathcal{V}}}\left(\sum_{k=1}^K \boldsymbol{d}^{l,k\prime}\boldsymbol{Y}^k(\boldsymbol{F}\hat{\boldsymbol{\zeta}}+\boldsymbol{g})\right) \leq b_l \quad \forall l \in [M] \\
& && \left(\boldsymbol{T}^0\boldsymbol{x} + \sum_{k=1}^K \boldsymbol{U}^k\boldsymbol{y}^{0,k} - \boldsymbol{v}^0\right) \\
& && \quad + \sum_{i=1}^N g_i\left(\boldsymbol{T}^i\boldsymbol{x} + \sum_{k=1}^K \boldsymbol{U}^k\boldsymbol{Y}^k\boldsymbol{e}^i - \boldsymbol{v}^i\right) = \boldsymbol{0} \\
& && \sum_{i=1}^N F_{ij}\left(\boldsymbol{T}^i\boldsymbol{x} + \sum_{k=1}^K \boldsymbol{U}^k\boldsymbol{Y}^k\boldsymbol{e}^i - \boldsymbol{v}^i\right) = \boldsymbol{0} \qquad \forall j \in [N_E] \\
& && \underline{\boldsymbol{y}}^k \leq \boldsymbol{y}^{0,k} + \boldsymbol{Y}^k(\boldsymbol{F}\boldsymbol{\zeta}+\boldsymbol{g}) \leq \overline{\boldsymbol{y}}^k \quad \forall \boldsymbol{\zeta}\in\mathcal{V} \qquad \forall k \in [K] \\
& && \boldsymbol{Y}^k\boldsymbol{F}\boldsymbol{e}^j = \boldsymbol{0} \qquad \forall j \notin \Phi_k, \forall k \in [K] \\
& && \boldsymbol{x} \geq \boldsymbol{0}.
\end{aligned}
$$

Since $\boldsymbol{M}(\cdot)$ represents a segregation, $\boldsymbol{F}$ is the horizontal concatenation of L identity matrices. Hence, $\forall j \in [N_E]$, $\boldsymbol{F}\boldsymbol{e}^j = \boldsymbol{e}^i$ where $(i-1) \equiv (j-1) \bmod N$. Notice that $\boldsymbol{e}^j \in \Re^{N_E}$ while $\boldsymbol{e}^i \in \Re^N$. In particular, for any index set $I \subseteq [N]$,

$$
\boldsymbol{Y}\boldsymbol{e}^i = \boldsymbol{0} \quad \forall i \in I \quad \Leftrightarrow \quad \boldsymbol{Y}\boldsymbol{F}\boldsymbol{e}^j = \boldsymbol{0} \quad \forall j \in \{j \in [N_E] : \exists i \in I : (i-1) \equiv (j-1) \bmod N\}.
$$

In particular, if we choose $I = I_k^c$, then using the definition of Φ_k (4.11), we can express the above as

$$
\boldsymbol{Y}\boldsymbol{e}^i = \boldsymbol{0} \quad \forall i \notin I_k \quad \Leftrightarrow \quad \boldsymbol{Y}\boldsymbol{F}\boldsymbol{e}^j = \boldsymbol{0} \quad \forall j \notin \Phi_k,
$$

by applying the definition of Φ_k (4.11), Hence, using (4.8), any feasible point $\left(\boldsymbol{x}, \left\{\boldsymbol{y}^{0,k}, \boldsymbol{Y}^k\right\}_{k=1}^K\right)$ in Problem (4.2) is also feasible in Problem (4.12), and since their objectives coincide, we have $Z_{SLDR} \leq Z_{LDR}$. ■

Appendix B Proofs of Bounds on $\mathrm{E}_{\mathbb{P}}\left((\cdot)^+\right)$

B.1 Proof of Theorem 1

For the case of a fixed mean $\hat{\boldsymbol{\zeta}} = \boldsymbol{\mu}$, Natarajan et al. [3, Theorem 2.2], provided a tight bound in for the expectation of a general piecewise-linear utility function applied to an LDR. We specialize their result for the case of the utility function $u(x) = x^+$, to obtain

$$\sup_{supp(\tilde{\boldsymbol{\zeta}})\subseteq\mathcal{V},\hat{\boldsymbol{\zeta}}=\boldsymbol{\mu}} \mathrm{E}_{\mathbb{P}}\left(\left(r^0 + \boldsymbol{r}'\tilde{\boldsymbol{\zeta}}\right)^+\right) \quad = \quad \inf_{\boldsymbol{s}\in\Re^{N_E}} \left(\boldsymbol{s}'\boldsymbol{\mu} + \sup_{\boldsymbol{\zeta}\in\mathcal{V}} \left(\max\left\{r^0 + \boldsymbol{r}'\boldsymbol{\zeta} - \boldsymbol{s}'\boldsymbol{\zeta}, -\boldsymbol{s}'\boldsymbol{\zeta}\right\}\right)\right),$$

and equality is obtained because of the strong duality result of Isii [2]. In general, if the mean is not fixed, the ambiguity-averse bound on $\mathrm{E}_{\mathbb{P}}\left(\left(r^0 + \boldsymbol{r}'\tilde{\boldsymbol{\zeta}}\right)^+\right)$ is simply obtained by taking the supremum over the allowed values of $\hat{\boldsymbol{\zeta}} \in \hat{\mathcal{V}}$, which yields

$$\sup_{\mathbb{P}\in\mathbb{F}_1} \mathrm{E}_{\mathbb{P}}\left(\left(r^0 + \boldsymbol{r}'\tilde{\boldsymbol{\zeta}}\right)^+\right) = \pi^1\left(r^0, \boldsymbol{r}\right),$$

as required. ■

B.2 Proof of Theorem 2

In Natarajan et al [3, Theorem 2.1], the authors use a projection method by Popescu [4, Theorem 1] to show that if $\tilde{\boldsymbol{\zeta}}$ has a known mean $\hat{\boldsymbol{\zeta}} = \boldsymbol{\mu}$ and covariance $\boldsymbol{\Sigma}_{\mathcal{V}}$, the following equality holds:

$$\sup_{\tilde{\boldsymbol{\zeta}}\sim(\boldsymbol{\mu},\boldsymbol{\Sigma}_{\mathcal{V}})} \mathrm{E}_{\mathbb{P}}\left(\left(r^0 + \boldsymbol{r}'\tilde{\boldsymbol{\zeta}}\right)^+\right) = \frac{1}{2}\left(r^0 + \boldsymbol{r}'\boldsymbol{\mu}\right) + \frac{1}{2}\sqrt{(r^0 + \boldsymbol{r}'\boldsymbol{\mu})^2 + \boldsymbol{r}'\boldsymbol{\Sigma}_{\mathcal{V}}\boldsymbol{r}}.$$

To construct the worst-case bounds of $\mathrm{E}_{\mathbb{P}}\left(\left(r^0 + \boldsymbol{r}'\tilde{\boldsymbol{\zeta}}\right)^+\right)$ over all $\mathbb{P} \in \mathbb{F}_2$, we simply need to find the supremum over all allowable $\left(\hat{\boldsymbol{\zeta}}, \boldsymbol{\Sigma}_{\mathcal{V}}\right)$ in $\mathbb{F}_2$. We obtain the bound by solving the following optimization problem:

$$\begin{aligned}\sup_{\mathbb{P}\in\mathbb{F}_2} \mathrm{E}_{\mathbb{P}}\left(\left(r^0 + \boldsymbol{r}'\tilde{\boldsymbol{\zeta}}\right)^+\right) = \eta^2(r^0, \boldsymbol{r}) \triangleq \quad & \sup_{\hat{\boldsymbol{\zeta}},\boldsymbol{\Sigma}_{\mathcal{V}}} \quad \left\{\frac{1}{2}\left(r^0 + \boldsymbol{r}'\hat{\boldsymbol{\zeta}}\right) + \frac{1}{2}\sqrt{(r^0 + \boldsymbol{r}'\hat{\boldsymbol{\zeta}})^2 + \boldsymbol{r}'\boldsymbol{\Sigma}_{\mathcal{V}}\boldsymbol{r}}\right\} \\ & \text{s.t.} \quad \boldsymbol{F}\boldsymbol{\Sigma}_{\mathcal{V}}\boldsymbol{F}' = \boldsymbol{\Sigma} \\ & \qquad\;\; \boldsymbol{\Sigma}_{\mathcal{V}} \in \mathbb{S}_+^{N_E} \\ & \qquad\;\; \hat{\boldsymbol{\zeta}} \in \hat{\mathcal{V}},\end{aligned}$$

where $\mathbb{S}_+^N$ denotes the positive semidefinite cone of symmetric $N\times N$ matrices. We complete the proof by showing that the bounds $\eta^2(r^0, \boldsymbol{r})$ and $\pi^2\left(r^0, \boldsymbol{r}\right)$ are equivalent. Suppose $\exists \boldsymbol{y} \in \Re^N$ such that $\boldsymbol{F}'\boldsymbol{y} = \boldsymbol{r}$, then $\boldsymbol{r}'\boldsymbol{\Sigma}_{\mathcal{V}}\boldsymbol{r} = \boldsymbol{y}'\boldsymbol{F}\boldsymbol{\Sigma}_{\mathcal{V}}\boldsymbol{F}'\boldsymbol{y} = \boldsymbol{y}'\boldsymbol{\Sigma}\boldsymbol{y}$, and the bounds are easily seen to be equivalent. Now suppose $\nexists \boldsymbol{y} \in \Re^N$ such that $\boldsymbol{F}'\boldsymbol{y} = \boldsymbol{r}$. This causes the outer optimization problem defining $\pi^2\left(r^0, \boldsymbol{r}\right)$ to be infeasible, and $\pi^2\left(r^0, \boldsymbol{r}\right) = +\infty$. We proceed to establish that $\eta^2(r^0, \boldsymbol{r}) = +\infty$ as well. To begin the proof, we choose

$$\boldsymbol{\Sigma}_{\mathcal{V}}^0 = \boldsymbol{F}'\left(\boldsymbol{F}\boldsymbol{F}'\right)^{-1}\boldsymbol{\Sigma}\left(\boldsymbol{F}\boldsymbol{F}'\right)^{-1}\boldsymbol{F},$$

which satisfies $\boldsymbol{F}\boldsymbol{\Sigma}_{\mathcal{V}}^{0}\boldsymbol{F}' = \boldsymbol{\Sigma}$. We notice that $\boldsymbol{F}\boldsymbol{F}'$ is invertible since $N_E \geq N$, and $\boldsymbol{F}$ is assumed to be full rank. Next, we choose $\boldsymbol{y} = (\boldsymbol{F}\boldsymbol{F}')^{-1}\boldsymbol{F}\boldsymbol{r}$, and express $\boldsymbol{r} = \boldsymbol{F}'\boldsymbol{y} + \boldsymbol{r}_{\perp}$. By assumption, $\boldsymbol{F}'\boldsymbol{y} \neq \boldsymbol{r}$, which implies $\boldsymbol{r}_{\perp} \neq \boldsymbol{0}$. Furthermore, we have

$$\begin{aligned} \boldsymbol{F}\boldsymbol{r}_{\perp} &= \boldsymbol{F}\boldsymbol{r} - \boldsymbol{F}\boldsymbol{F}'\boldsymbol{y} \\ &= \boldsymbol{F}\boldsymbol{r} - \boldsymbol{F}\boldsymbol{F}'(\boldsymbol{F}\boldsymbol{F}')^{-1}\boldsymbol{F}\boldsymbol{r} \\ &= \boldsymbol{0}. \end{aligned}$$

Now, for some $\lambda \in \Re_{+}$, consider

$$\boldsymbol{\Sigma}_{\mathcal{V}}(\lambda) = \boldsymbol{\Sigma}_{\mathcal{V}}^{0} + \lambda \boldsymbol{r}_{\perp}\boldsymbol{r}_{\perp}'.$$

We notice that $\boldsymbol{\Sigma}_{\mathcal{V}}(\lambda) \in \mathbb{S}_{+}^{N_E}$. Furthermore, we have

$$\begin{aligned} \boldsymbol{F}\Big(\boldsymbol{\Sigma}_{\mathcal{V}}(\lambda)\Big)\boldsymbol{F}' &= \boldsymbol{F}\boldsymbol{\Sigma}_{\mathcal{V}}^{0}\boldsymbol{F}' + \boldsymbol{0} \\ &= \boldsymbol{\Sigma}\ . \end{aligned}$$

Hence, for any $\hat{\boldsymbol{\zeta}} \in \hat{\mathcal{V}}$, $\lambda \in \Re_{+}$, $\eta^2\left(r^0, \boldsymbol{r}\right)$ is bounded from below by

$$\begin{aligned} \eta^2\left(r^0, \boldsymbol{r}\right) &\geq \frac{1}{2}\left(r^0 + \boldsymbol{r}'\hat{\boldsymbol{\zeta}}\right) + \frac{1}{2}\sqrt{(r^0 + \boldsymbol{r}'\hat{\boldsymbol{\zeta}})^2 + \boldsymbol{r}'\boldsymbol{\Sigma}_{\mathcal{V}}(\lambda)\boldsymbol{r}} \\ &= \frac{1}{2}\left(r^0 + \boldsymbol{r}'\hat{\boldsymbol{\zeta}}\right) + \frac{1}{2}\sqrt{(r^0 + \boldsymbol{r}'\hat{\boldsymbol{\zeta}})^2 + \boldsymbol{r}'\boldsymbol{\Sigma}_{\mathcal{V}}^{0}\boldsymbol{r} + \lambda\left(\boldsymbol{r}'\boldsymbol{r}_{\perp}\right)^2} \\ &= \frac{1}{2}\left(r^0 + \boldsymbol{r}'\hat{\boldsymbol{\zeta}}\right) + \frac{1}{2}\sqrt{(r^0 + \boldsymbol{r}'\hat{\boldsymbol{\zeta}})^2 + \boldsymbol{r}'\boldsymbol{\Sigma}_{\mathcal{V}}^{0}\boldsymbol{r} + \lambda\left(\boldsymbol{y}'\boldsymbol{F}\boldsymbol{r}_{\perp} + \boldsymbol{r}_{\perp}'\boldsymbol{r}_{\perp}\right)^2} \\ &= \frac{1}{2}\left(r^0 + \boldsymbol{r}'\hat{\boldsymbol{\zeta}}\right) + \frac{1}{2}\sqrt{(r^0 + \boldsymbol{r}'\hat{\boldsymbol{\zeta}})^2 + \boldsymbol{r}'\boldsymbol{\Sigma}_{\mathcal{V}}^{0}\boldsymbol{r} + \lambda\left\|\boldsymbol{r}_{\perp}\right\|_2^4}. \end{aligned}$$

Taking the limit as $\lambda \to \infty$, the lower bound (i.e. right-hand side) approaches $+\infty$. Thus, if $\nexists \boldsymbol{y}$ such that $\boldsymbol{F}'\boldsymbol{y} = \boldsymbol{r}$, then $\eta^2(r^0, \boldsymbol{r}) = +\infty$ as desired. ■

B.3 Proof of Theorem 3

We only have to prove the bound in the non-infinite case. We begin by noticing that we can express $x^0 + \boldsymbol{x}'\tilde{\boldsymbol{z}}_\sigma$ as:

$$x^0 + \boldsymbol{x}'\tilde{\boldsymbol{z}}_\sigma \equiv x^0 + \boldsymbol{x}'\hat{\boldsymbol{z}}_\sigma + \boldsymbol{x}'\left(\tilde{\boldsymbol{z}}_\sigma - \hat{\boldsymbol{z}}_\sigma\right).$$

Now, we use the property $\forall \lambda > 0$ that

$$w^+ \leq \frac{\lambda}{e}\exp\left(\frac{w}{\lambda}\right), \forall w \in \Re,$$

and the independence of each component of $\tilde{\boldsymbol{z}}_\sigma$ to obtain the general bound

$$\mathrm{E}_{\mathbb{P}}\left(\left(x^0 + \boldsymbol{x}'\tilde{\boldsymbol{z}}_\sigma\right)^+\right) \leq \frac{\lambda}{e}\exp\left(\frac{1}{\lambda}\sup_{\hat{\boldsymbol{z}}_\sigma \in \hat{\mathcal{W}}_\sigma}\left\{x^0 + \boldsymbol{x}'\hat{\boldsymbol{z}}_\sigma\right\}\right)\prod_{j=1}^{N_\sigma}\mathrm{E}_{\mathbb{P}}\left(\exp\left(\frac{x_j\left(\tilde{z}_{\sigma,j} - \hat{z}_{\sigma,j}\right)}{\lambda}\right)\right).$$

Now, using the definition of the forward and backward deviations in [1, Equations (17) and (18)], $\forall \mathbb{P} \in \mathbb{F}_3$,

$$\ln\left(\mathrm{E}_{\mathbb{P}}\left(\exp\left(\frac{x_j\left(\tilde{z}_{\sigma,j} - \hat{z}_{\sigma,j}\right)}{\lambda}\right)\right)\right) \leq \begin{cases} x_j^2\sigma_{f,j}^2/2\lambda^2 & \text{if } x_j \geq 0, \\ x_j^2\sigma_{b,j}^2/2\lambda^2 & \text{otherwise.} \end{cases}$$

Combining these results, when $(r^0, \boldsymbol{r}) = (x^0 + \boldsymbol{x}'\boldsymbol{g}_\sigma, \boldsymbol{F}'_\sigma \boldsymbol{x})$, we get

$$\sup_{\mathbb{P}\in\mathbb{F}_3} \mathrm{E}_\mathbb{P}\left(\left(r^0 + \boldsymbol{r}'\tilde{\boldsymbol{\zeta}}\right)^+\right) = \sup_{\mathbb{P}\in\mathbb{F}_3} \mathrm{E}_\mathbb{P}\left((x^0 + \boldsymbol{x}'\tilde{\boldsymbol{z}}_\sigma)^+\right) \le \psi(x^0, \boldsymbol{x}). \tag{B.1}$$

Next, using the identity $x^+ \equiv x + x^- \ \forall x \in \Re$, by the same argument,

$$\sup_{\mathbb{P}\in\mathbb{F}_3} \mathrm{E}_\mathbb{P}\left(\left(r^0 + \boldsymbol{r}'\tilde{\boldsymbol{\zeta}}\right)^+\right) \le \left(r^0 + \sup_{\hat{\boldsymbol{\zeta}}\in\hat{\mathcal{V}}} \boldsymbol{r}'\hat{\boldsymbol{\zeta}}\right) + \psi(-x^0, -\boldsymbol{x}). \tag{B.2}$$

Since choosing $(s^0, \boldsymbol{s}) = (x^0, \boldsymbol{x})$ in (5.4) reduces to (B.1) and choosing $(s^0, \boldsymbol{s}) = (0, \boldsymbol{0})$ in (5.4) reduces to (B.2), we have shown that $\pi^3\left(r^0, \boldsymbol{r}\right)$ is not larger than either (B.1) or (B.2). Finally, we establish that $\pi^3\left(r^0, \boldsymbol{r}\right)$ indeed bounds $\mathrm{E}_\mathbb{P}\left(\left(r^0 + \boldsymbol{r}'\tilde{\boldsymbol{\zeta}}\right)^+\right)$ from above. For any $(s^0, \boldsymbol{s}, x^0, \boldsymbol{x})$ such that $(x^0 + \boldsymbol{x}'\boldsymbol{g}_\sigma, \boldsymbol{F}'_\sigma \boldsymbol{x}) = (r^0, \boldsymbol{r})$, we have

$$\begin{aligned}
&(r^0 - s^0) - \boldsymbol{s}'\boldsymbol{g}_\sigma + \sup_{\hat{\boldsymbol{\zeta}}\in\hat{\mathcal{V}}} \left(\boldsymbol{r} - \boldsymbol{s}'\boldsymbol{F}_\sigma\right)\hat{\boldsymbol{\zeta}} + \psi(s^0 - x^0, \boldsymbol{s} - \boldsymbol{x}) + \psi(s^0, \boldsymbol{s}) \\
&\ge\ (x^0 - s^0) + (\boldsymbol{x} - \boldsymbol{s})'\boldsymbol{g}_\sigma + \sup_{\hat{\boldsymbol{\zeta}}\in\hat{\mathcal{V}}} \left((\boldsymbol{x} - \boldsymbol{s})'\boldsymbol{F}_\sigma\hat{\boldsymbol{\zeta}}\right) + \psi(s^0 - x^0, \boldsymbol{s} - \boldsymbol{x}) + \psi(s^0, \boldsymbol{s}) \\
&\ge\ \sup_{\mathbb{P}\in\mathbb{F}_3} \mathrm{E}_\mathbb{P}\left(\left((x^0 - s^0) + (\boldsymbol{x} - \boldsymbol{s})'\tilde{\boldsymbol{z}}_\sigma\right)^+\right) + \sup_{\mathbb{P}\in\mathbb{F}_3} \mathrm{E}_\mathbb{P}\left(\left(s^0 + \boldsymbol{s}'\tilde{\boldsymbol{z}}_\sigma\right)^+\right) \quad \text{(by (B.1) and (B.2))} \\
&\ge\ \sup_{\mathbb{P}\in\mathbb{F}_3} \mathrm{E}_\mathbb{P}\left(\left(x^0 + \boldsymbol{x}'\tilde{\boldsymbol{z}}_\sigma\right)^+\right) \quad \text{(by subadditivity)} \\
&=\ \sup_{\mathbb{P}\in\mathbb{F}_3} \mathrm{E}_\mathbb{P}\left(\left(r^0 + \boldsymbol{r}'\tilde{\boldsymbol{\zeta}}\right)^+\right).
\end{aligned}$$

Since the above inequality holds for *any* choice of $(s^0, \boldsymbol{s}, x^0, \boldsymbol{x})$ which satisfies $(x^0 + \boldsymbol{x}'\boldsymbol{g}_\sigma, \boldsymbol{F}'_\sigma \boldsymbol{x}) = (r^0, \boldsymbol{r})$, it also holds when we take the infimum, and hence $\pi^3\left(r^0, \boldsymbol{r}\right)$ bounds $\sup_{\mathbb{P}\in\mathbb{F}_3} \mathrm{E}_\mathbb{P}\left(\left(r^0 + \boldsymbol{r}'\tilde{\boldsymbol{\zeta}}\right)^+\right)$ from above as required. ■

B.4 Proof of Theorem 4

B.4.1 Lemma: Positive Homogeneity of $\pi^s\left(r^0, \boldsymbol{r}\right)$

Lemma 1 *The bounding functions $\pi^s\left(r^0, \boldsymbol{r}\right)$ are positively homogeneous for each $s \in \{1, 2, 3\}$.*

Proof : The bounding functions $\pi^1\left(r^0, \boldsymbol{r}\right)$ and $\pi^2\left(r^0, \boldsymbol{r}\right)$ are easily seen to be positive homogeneous. We shall only explicitly prove the positive homogeneity of $\pi^3\left(r^0, \boldsymbol{r}\right)$. For any $\mu > 0$, we notice that

$$\psi(\mu x^0, \mu\boldsymbol{x}) = \inf_{\lambda>0}\left\{\frac{\lambda}{e}\exp\left(\frac{\mu}{\lambda}\sup_{\hat{\boldsymbol{z}}_\sigma\in\hat{\mathcal{W}}_\sigma}\left\{x^0 + \boldsymbol{x}'\hat{\boldsymbol{z}}_\sigma\right\} + \frac{\mu^2\|\boldsymbol{u}\|_2^2}{2\lambda^2}\right)\right\},$$

from the positive homogeneity of the supremum and norm operators. Re-expressing the minimization problem in terms of a new variable, $\nu = \frac{\lambda}{\mu}$,

$$\begin{aligned}
\psi(\mu x^0, \mu\boldsymbol{x}) &= \inf_{\nu>0}\left\{\frac{\mu\nu}{e}\exp\left(\frac{1}{\nu}\sup_{\hat{\boldsymbol{z}}_\sigma\in\hat{\mathcal{W}}_\sigma}\left\{x^0 + \boldsymbol{x}'\hat{\boldsymbol{z}}_\sigma\right\} + \frac{\|\boldsymbol{u}\|_2^2}{2\nu^2}\right)\right\} \\
&= \mu\psi(x^0, \boldsymbol{x}),
\end{aligned}$$

where the final equality comes from the positive homogeneity of the infimum operator. Now we consider

$$\pi^3\left(\mu r^0, \mu \boldsymbol{r}\right) = \inf_{\substack{s^0,\boldsymbol{s},x^0,\boldsymbol{x}\\ x^0+\boldsymbol{x}'\boldsymbol{g}_\sigma=\mu r^0\\ \boldsymbol{F}_\sigma'\boldsymbol{x}=\mu\boldsymbol{r}}} \left\{ \begin{array}{lcl} (\mu r^0 - s^0 - \boldsymbol{s}'\boldsymbol{g}_\sigma) & + & \sup_{\hat{\boldsymbol{\zeta}}\in\hat{\mathcal{V}}}(\mu\boldsymbol{r}' - \boldsymbol{s}'\boldsymbol{F}_\sigma)\hat{\boldsymbol{\zeta}} \\ & + & \psi(s^0-x^0, \boldsymbol{s}-\boldsymbol{x}) + \psi(s^0,\boldsymbol{s}) \end{array} \right\}$$

and, using the same idea as before, express the minimization problem in terms of new variables $(q^0, \boldsymbol{q}) = \left(\frac{s^0}{\mu}, \frac{\boldsymbol{s}}{\mu}\right)$, and $(w^0, \boldsymbol{w}) = \left(\frac{x^0}{\mu}, \frac{\boldsymbol{x}}{\mu}\right)$, we get

$$\pi^3\left(\mu r^0, \mu \boldsymbol{r}\right) = \inf_{\substack{q^0,\boldsymbol{q},w^0,\boldsymbol{w}\\ w^0+\boldsymbol{w}'\boldsymbol{g}_\sigma=r^0\\ \boldsymbol{F}_\sigma'\boldsymbol{w}=\boldsymbol{r}}} \left\{ \begin{array}{lcl} \mu(r^0 - q^0 - \boldsymbol{q}'\boldsymbol{g}_\sigma) & + & \mu\sup_{\hat{\boldsymbol{\zeta}}\in\hat{\mathcal{V}}}(\boldsymbol{r}' - \boldsymbol{q}'\boldsymbol{F}_\sigma)\hat{\boldsymbol{\zeta}} \\ & + & \psi(\mu q^0-\mu w^0, \mu\boldsymbol{q}-\mu\boldsymbol{w}) + \psi(\mu q^0,\mu\boldsymbol{q}) \end{array} \right\}.$$

Using the positive homogeneity of the infimum operator and $\psi(x^0, \boldsymbol{x})$ established earlier, we obtain $\pi^3\left(\mu r^0, \mu\boldsymbol{r}\right) = \mu\pi^3\left(r^0, \boldsymbol{r}\right)\ \forall \mu > 0$.

We consider the case of $\mu = 0$ separately. We first notice that the $\boldsymbol{H}_\sigma$ is full rank by definition, since it represents a mapping to an uncertainty vector $\boldsymbol{z}_\sigma$ with stochastically independent components. Hence $\boldsymbol{F}_\sigma = \boldsymbol{H}_\sigma\boldsymbol{F}$ is also full rank, and from the constraints, $(r^0, \boldsymbol{r}) = (0, \boldsymbol{0})$ implies $(x^0, \boldsymbol{x}) = (0, \boldsymbol{0})$. Simplifying, we have

$$\pi^3(0, \boldsymbol{0}) = \inf_{s^0,\boldsymbol{s}} \left\{ (s^0 - \boldsymbol{s}'\boldsymbol{g}_\sigma) + \sup_{\hat{\boldsymbol{\zeta}}\in\hat{\mathcal{V}}}(-\boldsymbol{s}'\boldsymbol{F}_\sigma)\hat{\boldsymbol{\zeta}} + 2\psi(s^0, \boldsymbol{s}) \right\}.$$

We know that $\pi^3(0, \boldsymbol{0}) \geq 0$ due to the upper bound property (Theorem 3). Furthermore, substituting the feasible $(s^0, \boldsymbol{s}) = (0, \boldsymbol{0})$ in the inner expression, and noticing that $\psi(0, \boldsymbol{0}) = 0$, we get $\pi^3(0, \boldsymbol{0}) \leq 0$. Thus $\pi^3(0, \boldsymbol{0}) = 0$, and $\pi^3\left(r^0, \boldsymbol{r}\right)$ is positive homogeneous. ■

B.4.2 Proof of Theorem 4

We begin by noticing that each $\pi^s\left(r^0, \boldsymbol{r}\right)$ is convex and positive homogeneous (Lemma 1) in its arguments. From positive homogeneity of each $\pi^s\left(r^0, \boldsymbol{r}\right)$, we have $\pi^s\left(0, \boldsymbol{0}\right) = 0$, which gives us the second inequality of (5.6). To establish that $\pi\left(r^0, \boldsymbol{r}\right)$ does indeed bound $\sup_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\left(r_0 + \boldsymbol{r}'\tilde{\boldsymbol{\zeta}}\right)^+\right)$ from above, we have for each $\mathbb{P} \in \bigcap_{s\in S} \mathbb{F}_s$

$$\begin{array}{rcll} \sum_{s\in S} \pi^s\left(r^{0,s}, \boldsymbol{r}^s\right) & \geq & \sum_{s\in S} \mathrm{E}_{\mathbb{P}}\left(\left(r^{0,s} + \boldsymbol{r}^{s\prime}\tilde{\boldsymbol{\zeta}}\right)^+\right) & \text{(using } \mathbb{P}\in\mathbb{F}_s\text{)} \\ & = & \mathrm{E}_{\mathbb{P}}\left(\sum_{s\in S}\left(r^{0,s} + \boldsymbol{r}^{s\prime}\tilde{\boldsymbol{\zeta}}\right)^+\right) & \text{(by linearity)} \\ & \geq & \mathrm{E}_{\mathbb{P}}\left(\left(\sum_{s\in S} r^{0,s} + \boldsymbol{r}^{s\prime}\tilde{\boldsymbol{\zeta}}\right)^+\right) & \text{(by subadditivity of } (\cdot)^+\text{)} \\ & = & \mathrm{E}_{\mathbb{P}}\left(\left(r^0 + \boldsymbol{r}'\tilde{\boldsymbol{\zeta}}\right)^+\right) & \text{(from (5.5)).} \end{array}$$

■

Appendix C Proofs of BDLDR Properties

C.1 Proof of Proposition 4

C.1.1 Lemma: Bounding a portion of the BDLDR

For clarity of exposition, we begin the proof with the following lemma:

Lemma 2 *For each $j \in [m]$, the following inequality holds:*

$$\underline{y}_j \leq r_j(\tilde{\boldsymbol{\zeta}}) + \left(r_j(\tilde{\boldsymbol{\zeta}}) - \underline{y}_j\right)^- \mathbb{1}_{\{j \in \underline{J}^\circ\}} - \left(r_j(\tilde{\boldsymbol{\zeta}}) - \overline{y}_j\right)^+ \mathbb{1}_{\{j \in \overline{J}^\circ\}} \leq \overline{y}_j \ .$$

Proof : We divide the proof into 4 cases:

Case 1: When $j \in [m] \setminus \left(\underline{J}^\circ \cup \overline{J}^\circ\right)$, the result holds directly from Equation (6.14).

Case 2: When $j \in \underline{J}^\circ \setminus \overline{J}^\circ$: To prove the upper bound, it suffices to consider the case when $j \in \overline{J}$. Furthermore, by assumption, $j \notin \overline{J}^\circ$, we can apply the linear constraints of Equation (6.14), $r_j(\tilde{\boldsymbol{\zeta}}) \leq \overline{y}_j$. Together with the obvious $\underline{y}_j \leq \overline{y}_j$, we get

$$r_j(\tilde{\boldsymbol{\zeta}}) + \left(r_j(\tilde{\boldsymbol{\zeta}}) - \underline{y}_j\right)^- = \max\left\{r_j(\tilde{\boldsymbol{\zeta}}), \underline{y}_j\right\} \leq \overline{y}_j \ .$$

Thus proving the upper bound. The lower bound follows directly from

$$\underline{y}_j \leq \max\left\{r_j(\tilde{\boldsymbol{\zeta}}), \underline{y}_j\right\} .$$

Case 3: When $j \in \overline{J}^\circ \setminus \underline{J}^\circ$, the proof follows an identical argument to **Case 2**.

Case 4: When $j \in \overline{J}^\circ \cap \underline{J}^\circ$, we have

$$r_j(\tilde{\boldsymbol{\zeta}}) + \left(r_j(\tilde{\boldsymbol{\zeta}}) - \underline{y}_j\right)^- - \left(r_j(\tilde{\boldsymbol{\zeta}}) - \overline{y}_j\right)^+ = \max\left\{\min\left\{r_j(\tilde{\boldsymbol{\zeta}}), \overline{y}_j\right\}, \underline{y}_j\right\} ,$$

which directly satisfies both upper and lower bounds.

■

C.1.2 Proof of Proposition 4

We notice that statement *1* of the proposition follows directly from the feasibility of each $\overline{\boldsymbol{p}}^i$ and $\overline{\boldsymbol{q}}^i$ in (6.12) and (6.13). To prove statement *2* of the proposition, we consider the j^{th} component of the BDLDR, by considering the statement $\underline{y}_j \leq \hat{r}_j(\tilde{\boldsymbol{\zeta}}) \leq \overline{y}_j$, $\forall j \in [m]$. We notice that the BDLDR can be written in the following verbose form:

$$\begin{aligned} \hat{r}_j(\tilde{\boldsymbol{\zeta}}) = \Bigg(& r_j(\tilde{\boldsymbol{\zeta}}) + \left(r_j(\tilde{\boldsymbol{\zeta}}) - \underline{y}_j\right)^- \mathbb{1}_{\{j \in \underline{J}^\circ\}} - \left(r_j(\tilde{\boldsymbol{\zeta}}) - \overline{y}_j\right)^+ \mathbb{1}_{\{j \in \overline{J}^\circ\}} \Bigg) \\ & + \sum_{i \in \underline{J}^\circ \setminus \{j\}} \left(r_i(\tilde{\boldsymbol{\zeta}}) - \underline{y}_i\right)^- \overline{p}^i_j + \sum_{i \in \overline{J}^\circ \setminus \{j\}} \left(r_i(\tilde{\boldsymbol{\zeta}}) - \overline{y}_i\right)^+ \overline{q}^i_j \ . \end{aligned}$$

To prove the upper bound, it suffices to consider $j \in \overline{J}$. We notice that we have explicitly removed j from both summation terms, so in both sums, $i \neq j$. Furthermore, since we only sum over indices i such that $\bar{\boldsymbol{p}}^i$ and $\bar{\boldsymbol{q}}^i$ are feasible in (6.12) and (6.13) respectively and we have established that $j \in \overline{J} \setminus \{i\}$, we have $\bar{p}^i_j \leq 0$ and $\bar{q}^i_j \leq 0$. Finally, the upper bound of Lemma 2 establishes the upper bound in the proposition statement. The lower bound can be proven with an identical argument. ■

C.2 Proof of Proposition 5

To show the second inequality, we begin by noting that each feasible solution of (6.19) is feasible in (6.22). We further note that $\forall i \in \underline{J},\ \pi^1(-r^0_i + \underline{y}_i, -\boldsymbol{R'e}^i) = 0$, (see Remark in Section 5.1). Similarly, $\forall i \in \overline{J},\ \pi^1(r^0_i - \overline{y}_i, \boldsymbol{R'e}^i) = 0$. Since $\underline{J}^\circ_{D,R} \subseteq \underline{J}$ and $\overline{J}^\circ_{D,R} \subseteq \overline{J}$, and using the property that $\pi(\cdot) \leq \pi^1(\cdot)$, we obtain $Z_{DLDR} \leq Z_{SLDR}$.

To prove the first inequality, we consider the sub-problems for the BDLDR, (6.12) and (6.13) against the corresponding sub-problems for the DLDR, (6.20) and (6.21). We notice that they are identical, with the sole exception that the BDLDR sub-problems (6.12, 6.13) have one less inequality constraint compared with the DLDR counterparts (6.20, 6.21). In particular, whenever $i \in \overline{J}$, the first DLDR sub-problem (6.20) is always infeasible. Conversely, if $i \notin \overline{J}$, both the BDLDR sub-problem (6.12) and DLDR sub-problem (6.20) are identical. A similar relation holds for the second sub-problem. This leads to the following set relations:

$$\begin{aligned} \underline{J}^\circ_D &= \underline{J}^\circ \setminus \overline{J}, \\ \overline{J}^\circ_D &= \overline{J}^\circ \setminus \underline{J}, \end{aligned} \tag{C.1}$$

and relations for the optimal solutions to the sub-problems:

$$\begin{aligned} \bar{\boldsymbol{p}}^i_{\boldsymbol{D}} &= \bar{\boldsymbol{p}}^i \quad \forall i \in \underline{J}^\circ_D, \\ \bar{\boldsymbol{q}}^i_{\boldsymbol{D}} &= \bar{\boldsymbol{q}}^i \quad \forall i \in \overline{J}^\circ_D. \end{aligned} \tag{C.2}$$

Together, these imply the set relations for the reduced index sets:

$$\begin{aligned} \underline{J}^\circ_{D,R} &= \underline{J}^\circ_R \setminus \overline{J}, \\ \overline{J}^\circ_{D,R} &= \overline{J}^\circ_R \setminus \underline{J}. \end{aligned} \tag{C.3}$$

Hence, using these relations, any feasible solution of (6.22) is feasible in (6.18). Using a similar argument to the DLDR vs SLDR above, we can relate the objectives by $Z_{BDLDR} \leq Z_{DLDR}$. ■

C.3 Proof of Proposition 6

From the BDLDR definition (6.29), it is obvious that the k^{th} BDLDR, $\boldsymbol{r}^k(\tilde{\boldsymbol{\zeta}})$ has no dependency for any $\zeta_j,\ \forall j \in \Phi^c_k$. Hence, $\hat{\Phi}_k \subseteq \Phi_k$ follows directly. Now, assuming problem (4.12) is feasible, we denote a feasible set of SLDRs to the problem as $\{\boldsymbol{r}^k\}_{k=1}^K$. We notice that $\{\boldsymbol{r}^k\}_{k=1}^K$ lies within the feasible region of problem (6.28), and is an valid candidate to construct our DLDR. However, feasibility in problem (4.12) implies that the nonlinear terms in (6.29) vanish, giving us $\hat{\boldsymbol{r}}^k(\tilde{\boldsymbol{\zeta}}) = \boldsymbol{r}^k(\tilde{\boldsymbol{\zeta}})$. Hence it is necessary that their information index sets agree, i.e. $\hat{\Phi}_k = \Phi_k$. ■

C.4 Proof of Proposition 7

We consider:

$$
\begin{aligned}
&\sum_{k=1}^{K}\sum_{j\in N^{-}(k)}\left(\sum_{i\in \underline{J}_j^{\circ}}\left(r_i^j(\tilde{\boldsymbol{\zeta}})-\underline{y}_i^j\right)^{-}\boldsymbol{U}^k\boldsymbol{p}^{i,j,k}+\sum_{i\in \overline{J}_j^{\circ}}\left(r_i^j(\tilde{\boldsymbol{\zeta}})-\overline{y}_i^j\right)^{+}\boldsymbol{U}^k\boldsymbol{q}^{i,j,k}\right)\\
=&\sum_{k=1}^{K}\sum_{j=1}^{K}\left(\sum_{i\in \underline{J}_j^{\circ}}\left(r_i^j(\tilde{\boldsymbol{\zeta}})-\underline{y}_i^j\right)^{-}\boldsymbol{U}^k\boldsymbol{p}^{i,j,k}+\sum_{i\in \overline{J}_j^{\circ}}\left(r_i^j(\tilde{\boldsymbol{\zeta}})-\overline{y}_i^j\right)^{+}\boldsymbol{U}^k\boldsymbol{q}^{i,j,k}\right)\mathbb{1}_{\{j\in N^{-}(k)\}}\\
=&\sum_{j=1}^{K}\sum_{k=1}^{K}\left(\sum_{i\in \underline{J}_j^{\circ}}\left(r_i^j(\tilde{\boldsymbol{\zeta}})-\underline{y}_i^j\right)^{-}\boldsymbol{U}^k\boldsymbol{p}^{i,j,k}+\sum_{i\in \overline{J}_j^{\circ}}\left(r_i^j(\tilde{\boldsymbol{\zeta}})-\overline{y}_i^j\right)^{+}\boldsymbol{U}^k\boldsymbol{q}^{i,j,k}\right)\mathbb{1}_{\{k\in N^{+}(j)\}}\\
=&\sum_{j=1}^{K}\sum_{k\in N^{+}(j)}\left(\sum_{i\in \underline{J}_j^{\circ}}\left(r_i^j(\tilde{\boldsymbol{\zeta}})-\underline{y}_i^j\right)^{-}\boldsymbol{U}^k\boldsymbol{p}^{i,j,k}+\sum_{k\in N^{+}(j)}\sum_{i\in \overline{J}_j^{\circ}}\left(r_i^j(\tilde{\boldsymbol{\zeta}})-\overline{y}_i^j\right)^{+}\boldsymbol{U}^k\boldsymbol{q}^{i,j,k}\right)\\
=&\sum_{j=1}^{K}\left(\sum_{k\in N^{+}(j)}\sum_{i\in \underline{J}_j^{\circ}}\left(r_i^j(\tilde{\boldsymbol{\zeta}})-\underline{y}_i^j\right)^{-}\boldsymbol{U}^k\boldsymbol{p}^{i,j,k}+\sum_{k\in N^{+}(j)}\sum_{i\in \overline{J}_j^{\circ}}\left(r_i^j(\tilde{\boldsymbol{\zeta}})-\overline{y}_i^j\right)^{+}\boldsymbol{U}^k\boldsymbol{q}^{i,j,k}\right),
\end{aligned}
$$

where we reverse the order of summation in the second equality, and use property (6.24). Considering the final expression, we note that the polyhedral regions $P(i,j)$ and $Q(i,j)$ are non-empty for $i\in\underline{J}_j^{\circ}$ and $i\in\overline{J}_j^{\circ}$ respectively, allowing us to apply the set of constraints (6.25) and (6.26). Applying the first constraint in each constraint set, we notice that the first summation vanishes for each $i\in\underline{J}_j^{\circ}$, and the second summation vanishes for each $i\in\overline{J}_j^{\circ}$. This causes the entire expression above to vanish, implying the result in statement *1* of the proposition. We prove statement *2* of the proposition by establishing the upper and lower bounds component-wise. We consider the n^{th} component of the k^{th} BDLDR, and rewrite it in the more verbose form:

$$
\begin{aligned}
\hat{r}_n^k(\tilde{\boldsymbol{\zeta}}) &= r_n^k(\tilde{\boldsymbol{\zeta}}) + \sum_{j\in N^{-}(k)}\sum_{i\in \underline{J}_j^{\circ}}\left(r_i^j(\tilde{\boldsymbol{\zeta}})-\underline{y}_i^j\right)^{-}p_n^{i,j,k} + \sum_{j\in N^{-}(k)}\sum_{i\in \overline{J}_j^{\circ}}\left(r_i^j(\tilde{\boldsymbol{\zeta}})-\overline{y}_i^j\right)^{+}q_n^{i,j,k}\\
&= r_n^k(\tilde{\boldsymbol{\zeta}}) + \sum_{i\in \underline{J}_k^{\circ}}\left(r_i^k(\tilde{\boldsymbol{\zeta}})-\underline{y}_i^k\right)^{-}p_n^{i,k,k} + \sum_{j\in N^{-}(k)\setminus\{k\}}\sum_{i\in \underline{J}_j^{\circ}}\left(r_i^j(\tilde{\boldsymbol{\zeta}})-\underline{y}_i^j\right)^{-}p_n^{i,j,k}\\
&\quad + \sum_{i\in \overline{J}_k^{\circ}}\left(r_i^k(\tilde{\boldsymbol{\zeta}})-\overline{y}_i^k\right)^{+}q_n^{i,k,k} + \sum_{j\in N^{-}(k)\setminus\{k\}}\sum_{i\in \overline{J}_j^{\circ}}\left(r_i^j(\tilde{\boldsymbol{\zeta}})-\overline{y}_i^j\right)^{+}q_n^{i,j,k}.
\end{aligned}
$$

And extracting the $i = n$ term from the first and third sums, we get the final expression:

$$\begin{aligned}
\hat{r}_n^k(\tilde{\boldsymbol{\zeta}}) &= r_n^k(\tilde{\boldsymbol{\zeta}}) + \left(r_n^k(\tilde{\boldsymbol{\zeta}}) - \underline{y}_n^k\right)^- \mathbb{1}_{\{n \in \underline{J}_k^\circ\}} - \left(r_n^k(\tilde{\boldsymbol{\zeta}}) - \overline{y}_n^k\right)^+ \mathbb{1}_{\{n \in \overline{J}_k^\circ\}} \\
&+ \underbrace{\sum_{i \in \underline{J}_k^\circ \setminus \{n\}} \left(r_i^k(\tilde{\boldsymbol{\zeta}}) - \underline{y}_i^k\right)^- p_n^{i,k,k}}_{(A)} + \underbrace{\sum_{j \in N^-(k) \setminus \{k\}} \sum_{i \in \underline{J}_j^\circ} \left(r_i^j(\tilde{\boldsymbol{\zeta}}) - \underline{y}_i^j\right)^- p_n^{i,j,k}}_{(B)} \\
&+ \underbrace{\sum_{i \in \overline{J}_k^\circ \setminus \{n\}} \left(r_i^k(\tilde{\boldsymbol{\zeta}}) - \overline{y}_i^k\right)^+ q_n^{i,k,k}}_{(C)} + \underbrace{\sum_{j \in N^-(k) \setminus \{k\}} \sum_{i \in \overline{J}_j^\circ} \left(r_i^j(\tilde{\boldsymbol{\zeta}}) - \overline{y}_i^j\right)^+ q_n^{i,j,k}}_{(D)}.
\end{aligned}$$

To prove the upper bound of statement *2* of the proposition, it suffices to consider $n \in \overline{J}_k$. Again, in each of the four sums in the expression above, we sum over indices i which correspond to feasible instances of constraint sets (6.25) and (6.26), and hence we can apply these constraints. We consider the sums (A) – (D) in turn. For (A), we notice that since $i \neq n$, using the third inequality of (6.25), $p_n^{i,k,k} \leq 0$. For (B), we notice that similar to (6.24), we have

$$j \in N^-(k) \setminus \{k\} \Leftrightarrow k \in N^+(j) \setminus \{j\}.$$

Hence, using the second inequality of (6.25), $p_n^{i,j,k} \leq 0$ in (B). Also, using the first inequality of (6.26), $q_n^{i,k,k} \leq 0$ in (C) and $q_n^{i,j,k} \leq 0$ in (D). The upper bound follows directly using Lemma 2. The lower bound can be proven using an identical argument. ■

C.5 Proof of Proposition 8

Any SLDR solution to Problem (2.2) will take the form of Problem (4.7), using the support sets $\mathcal{V}$ and $\hat{\mathcal{V}}$ to approximate the exact supports $\mathcal{V}^*$ and $\hat{\mathcal{V}}^*$. We begin by noting that $\forall j \in [K]$, $i \in \underline{J}_j$ implies $\pi^1\left(-r_i^{0,j} + \underline{y}_i, -\boldsymbol{R}^{j\prime}\boldsymbol{e}^i\right) = 0$, and similarly $i \in \overline{J}_j$ implies $\pi^1\left(r_i^{0,j} - \overline{y}_i, \boldsymbol{R}^{j\prime}\boldsymbol{e}^i\right) = 0$. Now, in each of the two summation terms (in the objective and constraints), $j \in N^-(k) \subseteq [K]$, and $i \in \underline{J}_{l,j,k}^\circ \subseteq \underline{J}_j$ (in the first sum) or $i \in \overline{J}_{l,j,k}^\circ \subseteq \overline{J}_j$ (in the second sum), for some $l \in \{0\} \cup [M]$. We further note that $\pi(\cdot) \leq \pi^1(\cdot)$. Hence, any feasible solution of the approximated Problem (4.7) is feasible in (6.30), and since their objectives coincide, the objectives are related by $Z_{BDLDR} \leq Z_{SLDR}$ as desired. ■

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