Distributionally Robust Optimization

and its Tractable Approximations with ROME

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Outline

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Introduction

Framework

Segregated LDR

Bi-Deflected LDR

ROME

Conclusion

Distributionally Robust Optimization

- An approach toward optimization under uncertainty
  - Partially characterized distribution
  - Historically called minimax stochastic programming (Žáčková, 1966)
  - Motivation: in practice we will never know the actual uncertainty distribution
- Contrast with classical Robust Optimization (RO)
  - Uncertainties in RO characterized by uncertainty set (support)
  - Ben-Tal and Nemirovski (1998), Bertsimas and Sim (2004)

Research Objectives

- Construct an approximate solution framework for a class of DRO problems. Features:
  - Multi-stage decisions
  - Distribution-free, but partially characterized by support, moments, and directional deviations (Chen, Sim, and Sun, 2007)
  - More flexible decision rules compared to AARC (Ben-Tal, Goryashko, Guslitzer, and Nemirovski, 2004), aka LDRs.
- To build a software modeling tool to solve DRO problems within this framework (ROME)

Outline

Notation

- $\tilde{\boldsymbol{z}} \in \Re^N$: uncertainties with distribution $\mathbb{P} \in \mathbb{F}$.
- $\boldsymbol{x} \in \Re^n$: "here-and-now" decision variables.
- $\boldsymbol{y}^k(\tilde{\boldsymbol{z}}) \in \Re^{m_k}$: "wait-and-see" decision rules, $\forall k \in [K]$.
- Each decision rule need not depend on the full uncertainty vector. We introduce K information index sets, to capture the dependency structure, denoted by $I_k \subseteq [N], \forall k \in [K]$.

$$\boldsymbol{y}^k \in \mathcal{Y}(m_k, N, I_k) \quad \forall k \in [K]$$

$$\mathcal{Y}(m, N, I) \triangleq \left\{ \boldsymbol{f} : \Re^N \to \Re^m \ : \ \boldsymbol{f}\left(\boldsymbol{z} + \sum_{i \notin I} \lambda_i \boldsymbol{e}^i\right) = \boldsymbol{f}(\boldsymbol{z}), \forall \boldsymbol{\lambda} \in \Re^N \right\}$$

- E.g. $\begin{array}{c} I_1 = \{2,3\} \\ \boldsymbol{y}^1 \in \mathcal{Y}(m, N, I_1) \end{array} \implies \boldsymbol{y}^1(\tilde{\boldsymbol{z}}) = \boldsymbol{y}^1(\tilde{z}_2, \tilde{z}_3)$
- For multi-stage decisions with progressive information revelation, $I_1 \subseteq I_2 \subseteq \ldots \subseteq I_K$

Model of Uncertainty

- Actual uncertainty distribution $\mathbb{P}$ lies in a family of distributions $\mathbb{F}$
- Partially characterized by distributional properties:
  - Support $\mathcal{W}$: Tractable conic representable, full-dim.
  - Mean Support $\hat{\mathcal{W}}$: Tractable conic representable.
  - Covariance matrix $\boldsymbol{\Sigma}$: Positive definite
  - Upper bounds on Directional Deviations for stochastically independent components $(\boldsymbol{H}_{\sigma}, \boldsymbol{\sigma_f}, \boldsymbol{\sigma_b})$.

Formulation of General Model

$$
\begin{array}{lll}
{Z_{GEN}}^* = & & \\
\min\limits_{\boldsymbol{x},\{\boldsymbol{y}^k(\cdot)\}_{k=1}^K} & \boldsymbol{c}^{0\prime}\boldsymbol{x} + \sup\limits_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\sum_{k=1}^K \boldsymbol{d}^{0,k\prime}\boldsymbol{y}^k(\tilde{\boldsymbol{z}})\right) & \\
\text{s.t.} & \boldsymbol{c}^{l\prime}\boldsymbol{x} + \sup\limits_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\sum_{k=1}^K \boldsymbol{d}^{l,k\prime}\boldsymbol{y}^k(\tilde{\boldsymbol{z}})\right) \leq b_l & \forall l \in [M] \\
& \boldsymbol{T}(\tilde{\boldsymbol{z}})\boldsymbol{x} + \sum_{k=1}^K \boldsymbol{U}^k\boldsymbol{y}^k(\tilde{\boldsymbol{z}}) = \boldsymbol{v}(\tilde{\boldsymbol{z}}) & \\
& \underline{\boldsymbol{y}}^k \leq \boldsymbol{y}^k(\tilde{\boldsymbol{z}}) \leq \overline{\boldsymbol{y}}^k & \forall k \in [K] \\
& \boldsymbol{x} \geq \boldsymbol{0} & \\
& \boldsymbol{y}^k \in \mathcal{Y}(m_k, N, I_k) & \forall k \in [K]
\end{array}
$$

Linear Approximation of General Model

- General model is intractable.
  - How to choose recourse decisions?
- Following Ben-Tal et. al. (2004), we restrict ourselves to LDRs (or AARCs), where the recourse decisions are affinely dependent on the model uncertainties.
- Define the space of LDRs

$$\mathcal{L}(m,N,I) \triangleq \left\{ \boldsymbol{f} : \Re^N \to \Re^m : \exists \left(\boldsymbol{y}^0, \boldsymbol{Y}\right) \in \Re^m \times \Re^{m \times N} : \begin{array}{c} \boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{y}^0 + \boldsymbol{Y}\boldsymbol{z} \\ \boldsymbol{Y}\boldsymbol{e}^i = \boldsymbol{0}, \forall i \notin I \end{array} \right\}$$

- Use this to approximate $\mathcal{Y}(m,N,I)$.

LDR Approximation of General Model

$$
\begin{array}{lll}
Z_{LDR} = & & \\
\min\limits_{\boldsymbol{x}, \{\boldsymbol{y}^k(\cdot)\}_{k=1}^K} & \boldsymbol{c}^{0\prime}\boldsymbol{x} + \sup\limits_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\sum_{k=1}^K \boldsymbol{d}^{0,k\prime}\boldsymbol{y}^k(\tilde{\boldsymbol{z}})\right) & \\
\text{s.t.} & \boldsymbol{c}^{l\prime}\boldsymbol{x} + \sup\limits_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\sum_{k=1}^K \boldsymbol{d}^{l,k\prime}\boldsymbol{y}^k(\tilde{\boldsymbol{z}})\right) \leq b_l & \forall l \in [M] \\
& \boldsymbol{T}(\tilde{\boldsymbol{z}})\boldsymbol{x} + \sum_{k=1}^K \boldsymbol{U}^k \boldsymbol{y}^k(\tilde{\boldsymbol{z}}) = \boldsymbol{v}(\tilde{\boldsymbol{z}}) & \\
& \underline{\boldsymbol{y}}^k \leq \boldsymbol{y}^k(\tilde{\boldsymbol{z}}) \leq \overline{\boldsymbol{y}}^k & \forall k \in [K] \\
& \boldsymbol{x} \geq \boldsymbol{0} & \\
& \boldsymbol{y}^k \in \mathcal{L}(m_k, N, I_k) & \forall k \in [K]
\end{array}
$$

LDR Approx. of General Model (Explicit)

$$
\begin{array}{lll}
Z_{LDR} = & & \\
\min_{\boldsymbol{x}, \{\boldsymbol{y}^{0,k}, \boldsymbol{Y}^k\}_{k=1}^K} & \boldsymbol{c}^{0\prime}\boldsymbol{x} + \sum_{k=1}^K \boldsymbol{d}^{0,k\prime}\boldsymbol{y}^{0,k} + \sup_{\hat{\boldsymbol{z}} \in \hat{\mathcal{W}}} \left(\sum_{k=1}^K \boldsymbol{d}^{0,k\prime}\boldsymbol{Y}^k \hat{\boldsymbol{z}} \right) & \\
\text{s.t.} & \boldsymbol{c}^{l\prime}\boldsymbol{x} + \sum_{k=1}^K \boldsymbol{d}^{l,k\prime}\boldsymbol{y}^{0,k} + \sup_{\hat{\boldsymbol{z}} \in \hat{\mathcal{W}}} \left(\sum_{k=1}^K \boldsymbol{d}^{l,k\prime}\boldsymbol{Y}^k \hat{\boldsymbol{z}} \right) \leq b_l & \forall l \in [M] \\
& \boldsymbol{T}^0\boldsymbol{x} + \sum_{k=1}^K \boldsymbol{U}^k \boldsymbol{y}^{0,k} = \boldsymbol{v}^0 & \\
& \boldsymbol{T}^j\boldsymbol{x} + \sum_{k=1}^K \boldsymbol{U}^k \boldsymbol{Y}^k \boldsymbol{e}^j = \boldsymbol{v}^j & \forall j \in [N] \\
& \underline{\boldsymbol{y}}^k \leq \boldsymbol{y}^{0,k} + \boldsymbol{Y}^k \boldsymbol{z} \leq \overline{\boldsymbol{y}}^k \quad \forall \boldsymbol{z} \in \mathcal{W} & \forall k \in [K] \\
& \boldsymbol{Y}^k \boldsymbol{e}^j = \boldsymbol{0} & \forall j \notin I_k, \forall k \in [K] \\
& \boldsymbol{x} \geq \boldsymbol{0} &
\end{array}
$$

Is LDR Approximation too Conservative?

- LDR approximation of General Model is tractable, but is it too conservative?
- We will proceed by using the basic LDR approximation as a starting point, and aim to find other more complex decision rules which are better.

$$Z_{GEN}{}^{*} \leq ??? \leq ??? \leq Z_{LDR}$$

Outline

Segregating the Uncertainties

- Idea: to re-map the uncertainties into a higher-dimensional space, and apply an LDR on the new uncertainties, for more flexibility.
- E.g. for a scalar uncertainty, we can segregate into positive and negative parts (as was done by Chen, Sim, Sun, and Zhang, 2008).

$$\tilde{z} = \underbrace{\tilde{z}^+}_{\tilde{\zeta}_1} - \underbrace{\tilde{z}^-}_{\tilde{\zeta}_2}$$

- Can segregate more generally into distinct segments of the real line.

SLDR Approximation of General Model

$$
\begin{array}{lll}
Z_{SLDR} = & & \\
\min\limits_{\boldsymbol{x}, \left\{\boldsymbol{r}^{0,k}, \boldsymbol{R}^k\right\}_{k=1}^K} & \boldsymbol{c}^{0\prime}\boldsymbol{x} + \sum\limits_{k=1}^K \boldsymbol{d}^{0,k\prime}\boldsymbol{r}^{0,k} + \sup\limits_{\hat{\zeta} \in \hat{\mathcal{V}}} \left(\sum\limits_{k=1}^K \boldsymbol{d}^{0,k\prime}\boldsymbol{R}^k\hat{\boldsymbol{\zeta}} \right) & \\
\text{s.t.} & \boldsymbol{c}^{l\prime}\boldsymbol{x} + \sum\limits_{k=1}^K \boldsymbol{d}^{l,k\prime}\boldsymbol{r}^{0,k} + \sup\limits_{\hat{\zeta} \in \hat{\mathcal{V}}} \left(\sum\limits_{k=1}^K \boldsymbol{d}^{l,k\prime}\boldsymbol{R}^k\hat{\boldsymbol{\zeta}} \right) \le b_l & \forall l \in [M] \\
& \boldsymbol{\mathcal{T}}^0\boldsymbol{x} + \sum\limits_{k=1}^K \boldsymbol{U}^k\boldsymbol{r}^{0,k} = \boldsymbol{\nu}^0 & \\
& \boldsymbol{\mathcal{T}}^j\boldsymbol{x} + \sum\limits_{k=1}^K \boldsymbol{U}^k\boldsymbol{R}^k\boldsymbol{e}^j = \boldsymbol{\nu}^j & \forall j \in [N_E] \\
& \underline{\boldsymbol{y}}^k \le \boldsymbol{r}^{0,k} + \boldsymbol{R}^k\boldsymbol{\zeta} \le \overline{\boldsymbol{y}}^k \quad \forall \boldsymbol{\zeta} \in \mathcal{V} & \forall k \in [K] \\
& \boldsymbol{R}^k\boldsymbol{e}^j = \boldsymbol{0} & \forall j \notin \Phi_k, \forall k \in [K] \\
& \boldsymbol{x} \ge \boldsymbol{0} &
\end{array}
$$

Properties of SLDR

Theorem

For the general problem, $Z_{SLDR} \leq Z_{LDR}$

- Increases the size of the problem (and computational effort needed), but retains the linear structure.
- Can we do better?

$$Z_{GEN}{}^* \leq ??? \leq Z_{SLDR} \leq Z_{LDR}$$

- Will base further improvement on the SLDR.

Outline

Review of Deflected LDR (DLDR)

- Originally introduced by Chen, Sim, Sun, and Zhang (2008) to circumvent restrictiveness imposed by the LDR.
  - The structure of the set of constraints might allow some piecewise-linear decision rules to be used instead.
  - Apply bounds on expected positive part of an LDR to exploit piecewise-linearity.
- Bi-deflected LDR seeks to improve and extend the DLDR.
  - Improve: Reduce the objective even further.
  - Extend: Explore modeling scenarios (non-anticipativity, expectations in constraints) which the DLDR didn't handle.

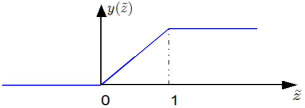
Motivating Example

- Scalar uncertainty $\tilde{z}$ with unknown distribution $\mathbb{P}$ in a family $\mathbb{F}$.
- $\mathbb{F}$ defined by infinite support, zero mean, and unit variance. Consider the following problem:

$$\begin{array}{ll} \min\limits_{y(\cdot)} & \sup\limits_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(|y(\tilde{z}) - \tilde{z}|\right) \\ & 0 \leq y(\tilde{z}) \leq 1 \\ & y \in \mathcal{Y}\left(1, 1, \{1\}\right) \end{array}$$

- Solution is pretty straightforward: piecewise-linear recourse function:

$$y_{sol}(\tilde{z}) = \begin{cases} \tilde{z} & \text{if } 0 \leq \tilde{z} \leq 1 \\ 0 & \text{if } \tilde{z} \leq 0 \\ 1 & \text{if } \tilde{z} \geq 1 \end{cases}$$



Example Reformulated into DRO Framework

- Apply the identities $x = x^+ - x^-, |x| = x^+ + x^-$.
- Model reformulates into DRO framework:

$$\begin{array}{cl} \min\limits_{y(\cdot)} & \sup\limits_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(|y(\tilde{z}) - \tilde{z}|\right) \\ & 0 \leq y(\tilde{z}) \leq 1 \\ & y \in \mathcal{Y}\left(1, 1, \{1\}\right) \end{array}$$

$$\Updownarrow$$

$$\begin{array}{cl} \min\limits_{y(\cdot),u(\cdot),v(\cdot)} & \sup\limits_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(u(\tilde{z}) + v(\tilde{z})\right) \\ \text{s.t.} & u(\tilde{z}) - v(\tilde{z}) = y(\tilde{z}) - \tilde{z} \\ & 0 \leq y(\tilde{z}) \leq 1 \\ & u(\tilde{z}), v(\tilde{z}) \geq 0 \\ & y, u, v \in \mathcal{Y}\left(1, 1, \{1\}\right) \end{array}$$

Using Different Decision Rules

- Using LDR, problem is infeasible, objective $= +\infty$.
- Using DLDR (of Chen et. al. 2008), we get the piecewise-linear decision rule:

$$\begin{aligned} \hat{u}_D(z) &= \left(u^0 + uz\right)^+ + \left(v^0 + vz\right)^- \\ \hat{v}_D(z) &= \left(v^0 + vz\right)^+ + \left(u^0 + uz\right)^- \\ y(z) &= y^0 + yz \end{aligned}$$

- After applying bounds, becomes an SOCP

$$\begin{aligned} Z_{DLDR} = \min_{u^0, u, v^0, v} \quad & \left\| \begin{pmatrix} u^0 \\ u \end{pmatrix} \right\|_2 + \left\| \begin{pmatrix} v^0 \\ v \end{pmatrix} \right\|_2 \\ \text{s.t.} \quad & u - v = -1 \\ & 0 \leq u^0 - v^0 \leq 1 \end{aligned}$$

- Objective $= 1$.

Different Decision Rules (II)

- Consider this hypothetical decision rule, which satisfies the model constraints:

$$\begin{aligned}\hat{u}(z) &= \left(u^0 + uz\right)^+ + \left(v^0 + vz\right)^- + \left(y^0 + yz\right)^- \\ \hat{v}(z) &= \left(v^0 + vz\right)^+ + \left(u^0 + uz\right)^- + \left(y^0 - 1 + yz\right)^+ \\ \hat{y}(z) &= \left(y^0 + yz\right)^+ - \left(y^0 - 1 + yz\right)^+\end{aligned}$$

- Applying robust bounds, problem transforms into the SOCP:

$$\begin{array}{rl}\min\limits_{y^0, y, u^0, u, v^0, v} & \left\|\begin{pmatrix} u^0 \\ u \end{pmatrix}\right\|_2 + \left\|\begin{pmatrix} v^0 \\ v \end{pmatrix}\right\|_2 + \frac{1}{2}\left\|\begin{pmatrix} y^0 \\ y \end{pmatrix}\right\|_2 + \frac{1}{2}\left\|\begin{pmatrix} y^0 - 1 \\ y \end{pmatrix}\right\|_2 - \frac{1}{2} \\ \text{s.t.} & u^0 - v^0 = y^0 \\ & u - v = y - 1\end{array}$$

- Objective $=$ 0.707 (better!)
- Aim to find algorithm to automatically predict this decision rule

Two-stage Bi-deflected LDR Problem Setup

- Consider a simpler two-stage model first:

$$
\begin{array}{rl}
\min\limits_{\boldsymbol{x},\boldsymbol{r}(\cdot)} & \boldsymbol{c}'\boldsymbol{x} + \sup\limits_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\boldsymbol{d}'\boldsymbol{r}(\tilde{\boldsymbol{\zeta}})\right) \\
\text{s.t.} & \boldsymbol{\mathcal{T}}(\tilde{\boldsymbol{\zeta}})\boldsymbol{x} + \boldsymbol{U}\boldsymbol{r}(\tilde{\boldsymbol{\zeta}}) = \boldsymbol{\nu}(\tilde{\boldsymbol{\zeta}}) \\
& \underline{\boldsymbol{y}} \leq \boldsymbol{r}(\tilde{\boldsymbol{\zeta}}) \leq \overline{\boldsymbol{y}} \\
& \boldsymbol{r} \in \mathcal{L}(m, N_E, [N_E])
\end{array}
$$

- Define the index sets of non-infinite bounds:

$$
\begin{aligned}
\underline{J} &= \left\{i \in [m] : \underline{y}_i > -\infty\right\} \\
\overline{J} &= \left\{i \in [m] : \overline{y}_i < +\infty\right\}
\end{aligned}
$$

Two-stage BDLDR Sub-problems

- We solve a series of sub-problems:

$$\underline{\forall i \in \underline{J}}$$

$$\begin{aligned} \min_{\boldsymbol{p}} \quad & \boldsymbol{d}'\boldsymbol{p} \\ \text{s.t.} \quad & \boldsymbol{U}\boldsymbol{p} = \boldsymbol{0} \\ & p_i = 1 \\ & p_j \geq 0 \qquad \forall j \in \underline{J} \\ & p_j \leq 0 \qquad \forall j \in \overline{J} \setminus \{i\} \end{aligned}$$

Feasible $i \in \underline{J}^\circ$

$$\underline{\forall i \in \overline{J}}$$

$$\begin{aligned} \min_{\boldsymbol{q}} \quad & \boldsymbol{d}'\boldsymbol{q} \\ \text{s.t.} \quad & \boldsymbol{U}\boldsymbol{q} = \boldsymbol{0} \\ & q_i = -1 \\ & q_j \leq 0 \qquad \forall j \in \overline{J} \\ & q_j \geq 0 \qquad \forall j \in \underline{J} \setminus \{i\} \end{aligned}$$

Feasible $i \in \overline{J}^\circ$

Two-stage BDLDR

- Consider an LDR which satisfies the relaxed constraints:

$$\begin{array}{ll} \boldsymbol{\mathcal{T}}(\tilde{\boldsymbol{\zeta}})\boldsymbol{x} + \boldsymbol{U}\boldsymbol{r}(\tilde{\boldsymbol{\zeta}}) = \boldsymbol{\nu}(\tilde{\boldsymbol{\zeta}}) & \\ r_j(\tilde{\boldsymbol{\zeta}}) \geq \underline{y}_j & \forall j \in \underline{J} \setminus \underline{J}^\circ \\ r_j(\tilde{\boldsymbol{\zeta}}) \leq \overline{y}_j & \forall j \in \overline{J} \setminus \overline{J}^\circ \end{array}$$

- The two-stage BDLDR is defined as:

$$\hat{\boldsymbol{r}}(\tilde{\boldsymbol{\zeta}}) \triangleq \underbrace{\boldsymbol{r}(\tilde{\boldsymbol{\zeta}})}_{\text{Linear Part}} + \sum_{i \in \underline{J}^\circ} \left(r_i(\tilde{\boldsymbol{\zeta}}) - \underline{y}_i \right)^- \underbrace{\bar{\boldsymbol{p}}^i} + \sum_{i \in \overline{J}^\circ} \left(r_i(\tilde{\boldsymbol{\zeta}}) - \overline{y}_i \right)^+ \underbrace{\bar{\boldsymbol{q}}^i}$$

Optimal solution of the i^{th} sub-problem (Deflected Part)

Two-stage BDLDR Properties

Theorem

The Bi-Deflected Linear Decision Rule, $\hat{\boldsymbol{r}}(\tilde{\boldsymbol{\zeta}})$, satisfies the following properties

1. $\boldsymbol{U}\hat{\boldsymbol{r}}(\tilde{\boldsymbol{\zeta}}) = \boldsymbol{U}\boldsymbol{r}(\tilde{\boldsymbol{\zeta}})$
2. $\underline{\boldsymbol{y}} \leq \hat{\boldsymbol{r}}(\tilde{\boldsymbol{\zeta}}) \leq \overline{\boldsymbol{y}}$

- For Reference:

$$\hat{\boldsymbol{r}}(\tilde{\boldsymbol{\zeta}}) \triangleq \underbrace{\boldsymbol{r}(\tilde{\boldsymbol{\zeta}})}_{\text{Linear Part}} + \sum_{i \in \underline{J}^\circ} \left(r_i(\tilde{\boldsymbol{\zeta}}) - \underline{y}_i\right)^- \underbrace{\bar{\boldsymbol{p}}^i} + \sum_{i \in \overline{J}^\circ} \left(r_i(\tilde{\boldsymbol{\zeta}}) - \overline{y}_i\right)^+ \underbrace{\bar{\boldsymbol{q}}^i}$$

Optimal solution of the i^{th} sub-problem (Deflected Part)

BDLDR Approximation of Two-stage Problem

- Using the BDLDR, the problem becomes:

$$
\begin{array}{rll}
Z_{BDLDR} = & & \\
\min\limits_{\boldsymbol{x},\boldsymbol{r}(\cdot)} & \boldsymbol{c}'\boldsymbol{x} + \sup\limits_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\boldsymbol{d}'\boldsymbol{r}(\tilde{\boldsymbol{\zeta}})\right) & \\
& + \sum\limits_{i\in\underline{J}^{\circ}_{R}} \sup\limits_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\left(r_i(\tilde{\boldsymbol{\zeta}}) - \underline{y}_i\right)^{-}\right)\boldsymbol{d}'\bar{\boldsymbol{p}}^{i} & \\
& + \sum\limits_{i\in\overline{J}^{\circ}_{R}} \sup\limits_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\left(r_i(\tilde{\boldsymbol{\zeta}}) - \overline{y}_i\right)^{+}\right)\boldsymbol{d}'\bar{\boldsymbol{q}}^{i} & \\
\text{s.t.} & & \\
& \boldsymbol{\mathcal{T}}(\tilde{\boldsymbol{\zeta}})\boldsymbol{x} + \boldsymbol{U}\boldsymbol{r}(\tilde{\boldsymbol{\zeta}}) = \boldsymbol{\nu}(\tilde{\boldsymbol{\zeta}}) & \\
& r_j(\tilde{\boldsymbol{\zeta}}) \geq \underline{y}_j & \forall j \in \underline{J} \setminus \underline{J}^{\circ} \\
& r_j(\tilde{\boldsymbol{\zeta}}) \leq \overline{y}_j & \forall j \in \overline{J} \setminus \overline{J}^{\circ} \\
& \boldsymbol{r} \in \mathcal{L}(m, N_E, [N_E]) &
\end{array}
$$

Performance of Two-stage BDLDR

Theorem

For the two-stage problem, $Z_{BDLDR} \leq Z_{DLDR} \leq Z_{LDR}$.

- Proof is a bit tedious, but the basic idea is that the BDLDR includes the DLDR as a special case, and hence is more flexible and therefore yields a lower objective.
- Next, to generalize the BDLDR for use together with non-anticipativity requirements and expectation constraints.

BDLDR for General Model

- Define the following sets $\forall k \in [K]$:

$$\begin{aligned} N^+(k) &= \{j \in [K] : \Phi_k \subseteq \Phi_j\} \\ N^-(k) &= \{j \in [K] : \Phi_j \subseteq \Phi_k\} \end{aligned}$$

- Similarly, we have two sub-problems:

$$\begin{array}{rcll} \multicolumn{4}{l}{\underline{\forall k \in [K], \forall i \in \underline{J}_k}} \\ \sum_{j \in N^+(k)} \boldsymbol{U}^j \boldsymbol{p}^{i,k,j} &=& \boldsymbol{0} & \\ p_l^{i,k,j} &\geq& 0 \quad \forall l \in \underline{J}_j & \forall j \in N^+(k) \\ p_l^{i,k,j} &\leq& 0 \quad \forall l \in \overline{J}_j & \forall j \in N^+(k) \setminus \{k\} \\ p_l^{i,k,k} &\leq& 0 \quad \forall l \in \overline{J}_k \setminus \{i\} & \\ p_i^{i,k,k} &=& 1 & \end{array}$$

$$\begin{array}{rcll} \multicolumn{4}{l}{\underline{\forall k \in [K], \forall i \in \overline{J}_k}} \\ \sum_{j \in N^+(k)} \boldsymbol{U}^j \boldsymbol{q}^{i,k,j} &=& \boldsymbol{0} & \\ q_l^{i,k,j} &\leq& 0 \quad \forall l \in \overline{J}_j & \forall j \in N^+(k) \\ q_l^{i,k,j} &\geq& 0 \quad \forall l \in \underline{J}_j & \forall j \in N^+(k) \setminus \{k\} \\ q_l^{i,k,k} &\geq& 0 \quad \forall l \in \underline{J}_k \setminus \{i\} & \\ q_i^{i,k,k} &=& -1 & \end{array}$$

Constructing the BDLDR

- Consider an LDR which satisfies the relaxed constraints

$$
\begin{aligned}
&\boldsymbol{\mathcal{T}}(\tilde{\boldsymbol{\zeta}})\boldsymbol{x} + \sum_{k=1}^{K} \boldsymbol{U}^k \boldsymbol{r}^k(\tilde{\boldsymbol{\zeta}}) = \boldsymbol{\nu}(\tilde{\boldsymbol{\zeta}}) && \\
&r_j^k(\tilde{\boldsymbol{\zeta}}) \geq \underline{y}_j^k && \forall k \in [K], \forall j \in \underline{J}_k \setminus \underline{J}_k^\circ \\
&r_j^k(\tilde{\boldsymbol{\zeta}}) \leq \overline{y}_j^k && \forall k \in [K], \forall j \in \overline{J}_k \setminus \overline{J}_k^\circ \\
&\boldsymbol{r}^k \in \mathcal{L}(m_k, N_E, \Phi_k) && \forall k \in [K]
\end{aligned}
$$

- The non-anticipative BDLDR is defined as:

$$
\hat{\boldsymbol{r}}^k(\tilde{\boldsymbol{\zeta}}) \triangleq \underbrace{\boldsymbol{r}^k(\tilde{\boldsymbol{\zeta}})}_{\text{Linear Part}} + \sum_{j \in N^-(k)} \Bigg(\sum_{i \in \underline{J}_j^\circ} \left(r_i^j(\tilde{\boldsymbol{\zeta}}) - \underline{y}_i^j \right)^- \underbrace{\boldsymbol{p}^{i,j,k}} + \sum_{i \in \overline{J}_j^\circ} \left(r_i^j(\tilde{\boldsymbol{\zeta}}) - \overline{y}_i^j \right)^+ \underbrace{\boldsymbol{q}^{i,j,k}} \Bigg)
$$

Optimal solution of i^{th} sub-problem (Deflected Part)

BDLDR Properties

Theorem

Each non-anticipative BDLDR, $\hat{\boldsymbol{r}}^k(\tilde{\boldsymbol{\zeta}})$ satisfies the following properties:

1. $\sum_{k=1}^{K} \boldsymbol{U}^k \hat{\boldsymbol{r}}^k(\tilde{\boldsymbol{\zeta}}) = \sum_{k=1}^{K} \boldsymbol{U}^k \boldsymbol{r}^k(\tilde{\boldsymbol{\zeta}})$
2. $\underline{\boldsymbol{y}}^k \leq \hat{\boldsymbol{r}}^k(\tilde{\boldsymbol{\zeta}}) \leq \overline{\boldsymbol{y}}^k, \quad \forall k \in [K]$

- For reference:

$$\hat{\boldsymbol{r}}^k(\tilde{\boldsymbol{\zeta}}) \triangleq \underbrace{\boldsymbol{r}^k(\tilde{\boldsymbol{\zeta}})}_{\text{Linear Part}} + \sum_{j \in N^-(k)} \underbrace{\left(\sum_{i \in \underline{J}_j^\circ} \left(r_i^j(\tilde{\boldsymbol{\zeta}}) - \underline{y}_i^j \right)^- \underbrace{\boldsymbol{p}^{i,j,k}} + \sum_{i \in \overline{J}_j^\circ} \left(r_i^j(\tilde{\boldsymbol{\zeta}}) - \overline{y}_i^j \right)^+ \underbrace{\boldsymbol{q}^{i,j,k}} \right)}_{\text{Optimal solution of } i^{th} \text{ sub-problem (Deflected Part)}}$$

BDLDR Performance

Theorem

For the general problem, we have

$$Z_{GEN}{}^{*} \leq Z_{BDLDR} \leq Z_{SLDR} \leq Z_{LDR}$$

.

- The k^{th} BDLDR sums over $j \in N^{-}(k)$,
  - These are the indices which are contained by the current (k^{th}) information index set,
  - BDLDR does not violate non-anticipative of the LDR which it is based upon.

Outline

Designing Software for RO Problems

- RO programs, especially with more complex decision rules (e.g. BDLDR), can be messy, typically involving the following steps:
  - Constructing robust counterparts,
  - Finding deflected components (non-anticipative),
  - Constructing robust bounds.
- Software Design Goals:
  - Mathematically intuitive modeling of RO problems
  - Rapid prototyping of RO ideas
  - Ease transition from RO theory to practice
  - Allow numerical studies

Robust Optimization Made Easy with ROME

- ROME: an algebraic modeling language in the MATLAB environment for modeling Robust Optimization (RO) problems
- ROME is primarily designed for structured RO problems, within the DRO framework presented.
  - Other MATLAB-based modeling languages (e.g. CVX, YALMIP) solve more general types of problems.
  - ROME focuses on modeling phenonena more specific to RO, e.g. uncertainty description, decision rules, non-anticipativity.
- ROME is a modeling language, does not do actual solving.
  - Calls underlying solver engines to do perform solve step
  - Present version supports CPLEX, MOSEK, and SDPT3 solver engines

Example: Robust Inventory Model

- Model a distribution-free, multi-period, inventory control problem with service constraints.
- Demand is exogenous, with unknown, but partially characterized distribution in a family $\mathbb{F}$, defined by:
  - Covariance Matrix: Temporal demand correlation can be modeled by a non-diagonal covariance matrix.
  - Mean: Assume fixed mean (for simplicity).
  - Support: Maximum demand in each period.
- Backorders allowed, but in some applications, a penalty cost might not be a good model for stockouts.
  - In our model, we avoid stockouts with a constraint on the fill-rate.
  - Fill-rate constraint acts as a service guarantee to the consumers.

Model Parameters

Parameters

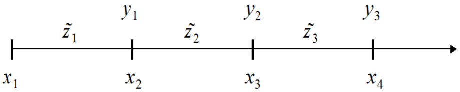
Num Periods : $T \in \mathbb{N}$

Order Cost : $\boldsymbol{c} \in \Re^T$

Holding Cost : $\boldsymbol{h} \in \Re^T$

Min Fill Rate : $\boldsymbol{\beta} \in \Re^T$

Max Order Qty : $\boldsymbol{x}^{MAX} \in \Re^T$



Uncertainties

Demand : $\tilde{\boldsymbol{z}} \in \Re^T$

Decisions

Order Quantity : $\boldsymbol{x}(\tilde{\boldsymbol{z}}) : x_t(\tilde{\boldsymbol{z}}) \in \mathcal{L}(1, T, [t-1])$

Inventory Level : $\boldsymbol{y}(\tilde{\boldsymbol{z}}) : y_t(\tilde{\boldsymbol{z}}) \in \mathcal{L}(1, T, [t])$

Robust Fill Rate Constraint

$$\text{Fill Rate} = \frac{\text{Expected Sales}}{\text{Expected Demand}} \geq \beta$$

- Using our notation, the robust (worst-case) version:

$$\inf_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}} \left(\min\left\{\tilde{z}_t, y_{t-1}(\tilde{\boldsymbol{z}}) + x_t(\tilde{\boldsymbol{z}})\right\}\right) \geq \beta_t \mu_t$$

- Apply inventory balance equation and re-arrange:

$$\sup_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}} \left(y_t(\tilde{\boldsymbol{z}})^-\right) \leq (1-\beta_t)\mu_t$$

Model Formulation

- Family of uncertainties:

$$\mathbb{F} = \left\{ \mathbb{P} : \mathrm{E}_{\mathbb{P}}\left(\tilde{\boldsymbol{z}}\right) = \boldsymbol{\mu}, \mathrm{E}_{\mathbb{P}}\left(\tilde{\boldsymbol{z}}\tilde{\boldsymbol{z}}'\right) = \boldsymbol{\Sigma} + \boldsymbol{\mu}\boldsymbol{\mu}', \mathbb{P}\left(0 \leq \tilde{\boldsymbol{z}} \leq \boldsymbol{z}^{MAX}\right) = 1 \right\}$$

- Robust Inventory Model with Fill Rate constraints:

$$\begin{array}{ll} \displaystyle\min_{\boldsymbol{x}(\cdot),\boldsymbol{y}(\cdot)} & \displaystyle\sup_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\boldsymbol{c}'\boldsymbol{x}(\tilde{\boldsymbol{z}}) + \boldsymbol{h}'\left(\boldsymbol{y}(\tilde{\boldsymbol{z}})\right)^{+}\right) \\ \text{s.t.} & \displaystyle\sup_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\boldsymbol{y}(\tilde{\boldsymbol{z}})^{-}\right) \leq (\boldsymbol{I} - \textbf{diag}\,(\boldsymbol{\beta}))\boldsymbol{\mu} \\ & \boldsymbol{D}\boldsymbol{y}\left(\tilde{\boldsymbol{z}}\right) - \boldsymbol{x}\left(\tilde{\boldsymbol{z}}\right) = -\tilde{\boldsymbol{z}} \\ & \boldsymbol{0} \leq \boldsymbol{x}\left(\tilde{\boldsymbol{z}}\right) \leq \boldsymbol{x}^{MAX} \\ & x_t \in \mathcal{L}(1, T, [t-1]) \quad \forall t \in [T] \\ & y_t \in \mathcal{L}(1, T, [t]) \quad \forall t \in [T] \end{array}$$

$$\boldsymbol{D} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ -1 & 1 & 0 & \dots & 0 \\ 0 & -1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

Model Transformed into DRO Framework

- After linearizing piecewise-linear terms,

$$\begin{array}{lll}
\min\limits_{\boldsymbol{x}(\cdot),\boldsymbol{y}(\cdot),\boldsymbol{r}(\cdot),\boldsymbol{s}(\cdot)} & \sup\limits_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\boldsymbol{c}'\boldsymbol{x}(\tilde{\boldsymbol{z}}) + \boldsymbol{h}'\boldsymbol{r}(\tilde{\boldsymbol{z}})\right) & \\
\text{s.t.} & \sup\limits_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\boldsymbol{s}(\tilde{\boldsymbol{z}})\right) \leq (\boldsymbol{I} - \mathbf{diag}\,(\boldsymbol{\beta}))\boldsymbol{\mu} & \\
& \boldsymbol{r}\left(\tilde{\boldsymbol{z}}\right) \geq \boldsymbol{y}\left(\tilde{\boldsymbol{z}}\right) & \\
& \boldsymbol{s}\left(\tilde{\boldsymbol{z}}\right) \geq -\boldsymbol{y}\left(\tilde{\boldsymbol{z}}\right) & \\
& \boldsymbol{r}\left(\tilde{\boldsymbol{z}}\right), \boldsymbol{s}\left(\tilde{\boldsymbol{z}}\right) \geq \mathbf{0} & \\
& \boldsymbol{D}\boldsymbol{y}\left(\tilde{\boldsymbol{z}}\right) - \boldsymbol{x}\left(\tilde{\boldsymbol{z}}\right) = -\tilde{\boldsymbol{z}} & \\
& \mathbf{0} \leq \boldsymbol{x}\left(\tilde{\boldsymbol{z}}\right) \leq \boldsymbol{x}^{MAX} & \\
& x_t \in \mathcal{L}(1, T, [t-1]) & \forall t \in [T] \\
& r_t, s_t, y_t \in \mathcal{L}(1, T, [t]) & \forall t \in [T]
\end{array}$$

ROME Code for Inventory Problem

```
% model parameters
T = 10;                     % planning horizon
c = 1 *ones(T, 1);          % order cost rate
hcost = 2*ones(T, 1);       % holding cost rate
beta  = 0.50*ones(T, 1); % minimum fillrate in each period
xMax  = 100*ones(T, 1);  % maximum order quantity in each period
alpha = 0.5;                % temporal autocorrelation factor
L = alpha * tril(ones(T), -1) + eye(T); % autocorrelation matrix

% numerical uncertainty parameters
zMax  = 105*ones(T, 1);   % maximum demand in each period
zMean = 30*ones(T, 1);    % mean demand in each period
zCovar = 20*(L * L');     % temporal demand covariance

% differencing matrix
D = eye(T) - diag(ones(T-1, 1), -1);

% dependency structure
pX = logical([tril(ones(T)), zeros(T, 1)]);
```

ROME Code for Inventory Problem (cont.)

```
% Step 3: BDLDR Method
% --------------------
h = rome_begin('Robust Inventory (BDLDR)');
tic;

% declare uncertainties
newvar z(T) uncertain nonneg;

% define uncertainty parameters
rome_constraint(z <= zMax);   % support
z.set_mean(zMean);            % mean
z.Covar = zCovar;             % covariance
```

$$\mathbb{F} = \left\{ \mathbb{P} : \begin{array}{l} \mathbb{P}\left(0 \leq \tilde{\boldsymbol{z}} \leq \boldsymbol{z}^{MAX}\right) = 1 \\ \mathrm{E}_{\mathbb{P}}\left(\tilde{\boldsymbol{z}}\right) = \boldsymbol{\mu}, \\ \mathrm{E}_{\mathbb{P}}\left(\tilde{\boldsymbol{z}}\tilde{\boldsymbol{z}}'\right) = \boldsymbol{\Sigma} + \boldsymbol{\mu}\boldsymbol{\mu}' \end{array} \right\}$$

```
% define LDR variables
newvar x(T, z, 'Pattern', pX) linearrule;  % order quantity
newvar y(T, z) linearrule;                 % inventory level

% define auxilliary variables
newvar r(T, z) s(T, z) linearrule nonneg;
```

$$\begin{array}{ll} \boldsymbol{r}\left(\tilde{\boldsymbol{z}}\right), \boldsymbol{s}\left(\tilde{\boldsymbol{z}}\right) \geq \boldsymbol{0} & \\ x_t \in \mathcal{L}(1, T, [t-1]) & \forall t \in [T] \\ r_t, s_t, y_t \in \mathcal{L}(1, T, [t]) & \forall t \in [T] \end{array}$$

ROME Code for Inventory Problem (cont.)

```
% auxilliary constraints
rome_constraint(r >=  y); % since r >= y^+
rome_constraint(s >= -y); % since s >= y^-

% fillrate constraint
rome_constraint(mean(s) <= diag(ones(T, 1) - beta) * zMean);

% inventory balance constraint
rome_constraint(D*y == x - z);

% order quantity constraints
rome_box(x, 0, xMax);

% objective
rome_minimize(c'*mean(x) + hcost'*mean(r));

% solve and display optimal objective
h.solve_deflected;
disp(sprintf('BDLDR Obj = %0.2f, time = %0.2f secs', h.ObjVal, toc));
x_sol_bdldr = h.eval(x)

rome_end;
```

$$\begin{array}{cl} \min\limits_{\boldsymbol{x}(\cdot),\boldsymbol{y}(\cdot),\boldsymbol{r}(\cdot),\boldsymbol{s}(\cdot)} & \sup\limits_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\boldsymbol{c}'\boldsymbol{x}(\tilde{\boldsymbol{z}}) + \boldsymbol{h}'\boldsymbol{r}(\tilde{\boldsymbol{z}})\right) \\ \text{s.t.} & \sup\limits_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\boldsymbol{s}(\tilde{\boldsymbol{z}})\right) \leq (\boldsymbol{I} - \mathbf{diag}\,(\boldsymbol{\beta}))\boldsymbol{\mu} \\ & \boldsymbol{D}\boldsymbol{y}\,(\tilde{\boldsymbol{z}}) - \boldsymbol{x}\,(\tilde{\boldsymbol{z}}) = -\tilde{\boldsymbol{z}} \\ & \boldsymbol{r}\,(\tilde{\boldsymbol{z}}) \geq \boldsymbol{y}\,(\tilde{\boldsymbol{z}}) \\ & \boldsymbol{s}\,(\tilde{\boldsymbol{z}}) \geq -\boldsymbol{y}\,(\tilde{\boldsymbol{z}}) \\ & \boldsymbol{0} \leq \boldsymbol{x}\,(\tilde{\boldsymbol{z}}) \leq \boldsymbol{x}^{MAX} \end{array}$$

ROME Output (MOSEK Solver)

```
EDU>> inventory_fillrate_example
Status: OPTIMAL
LDR Obj = 1747.50, time = 0.25 secs

x_sol_ldr =

 97.500
 0.000 + 0.952*z1
-0.000 + 0.007*z1 + 0.945*z2
 0.000 + 0.005*z1 + 0.010*z2 + 0.937*z3
-0.000 + 0.004*z1 + 0.006*z2 + 0.013*z3 + 0.929*z4
 0.000 + 0.003*z1 + 0.005*z2 + 0.008*z3 + 0.015*z4 + 0.921*z5
-0.000 + 0.004*z1 + 0.004*z2 + 0.006*z3 + 0.009*z4 + 0.018*z5 + 0.911*z6
-0.000 + 0.003*z1 + 0.005*z2 + 0.006*z3 + 0.008*z4 + 0.012*z5 + 0.023*z6 + 0.897*z7
 0.000 + 0.004*z1 + 0.005*z2 + 0.006*z3 + 0.009*z4 + 0.011*z5 + 0.016*z6 + 0.031*z7 + 0.870*z8
-0.000 + 0.007*z1 + 0.007*z2 + 0.009*z3 + 0.011*z4 + 0.015*z5 + 0.019*z6 + 0.031*z7 + 0.058*z8 + 0.795*z9

Status: NEAR_OPTIMAL
BDLDR Obj = 300.48, time = 0.58 secs

x_sol_bdldr =

 15.333
-0.000 + 1.003*z1 + 1.00(-0.000 + 1.003*z1)^- - 1.00(100.000 - 1.003*z1)^-
-0.000 + 0.000*z1 + 1.004*z2 + 1.00(-0.000 + 0.000*z1 + 1.004*z2)^- - 1.00(100.000 - 0.000*z1 - 1.004*z2)^-
-0.000 + 0.000*z1 - 0.000*z2 + 1.004*z3 + 1.00(-0.000 + 0.000*z1 - 0.000*z2 + 1.004*z3)^- - 1.00(100.000 - 0.0(
-0.000 - 0.000*z1 - 0.000*z2 + 0.000*z3 + 1.004*z4 + 1.00(-0.000 - 0.000*z1 - 0.000*z2 + 0.000*z3 + 1.004*z4)^-
 0.000 - 0.000*z1 - 0.000*z2 - 0.000*z3 + 0.000*z4 + 1.004*z5 + 1.00(0.000 - 0.000*z1 - 0.000*z2 - 0.000*z3 + (
-0.000 + 0.000*z1 - 0.000*z2 + 0.000*z3 + 0.000*z4 - 0.000*z5 + 1.004*z6 + 1.00(-0.000 + 0.000*z1 - 0.000*z2 +
 0.000 - 0.000*z1 - 0.000*z2 + 0.000*z3 + 0.000*z4 - 0.000*z5 - 0.000*z6 + 1.004*z7 + 1.00(0.000 - 0.000*z1 - (
-0.000 + 0.000*z1 - 0.000*z2 + 0.000*z3 + 0.000*z4 + 0.000*z5 - 0.000*z6 - 0.000*z7 + 1.004*z8 + 1.00(-0.000 +
-0.000 + 0.000*z1 + 0.000*z2 - 0.000*z3 + 0.000*z4 + 0.000*z5 + 0.000*z6 + 0.000*z7 + 0.000*z8 + 1.005*z9 + 1.(
```

Outline

Summary

- Different decision rules for Distributionally Robust Optimization:
  - LDRs: Tractable approximation, but poor performance
  - SLDRs: Improves performance, and retains linear structure
  - BDLDRs: Improves performance beyond SLDR, but makes problem messy
- ROME: Robust Optimization Made Easy
  - MATLAB-based modeling language for Distributionally Robust Optimization problems
  - Aims to allow robust optimization problems to be modeled and solved in a mathematically intuitive way
  - Freely-available for academic use from www.RobustOpt.com (together with User's Guide)