Distributionally Robust Optimization with ROME (part 1)

Joel Goh Melvyn Sim

Department of Decision Sciences
NUS Business School, Singapore

18 Jun 2009
NUS Business School
Guest Lecture

Outline

Outline

Classical Robust Optimization

- Robust optimization (RO) has been traditionally used to immunize an uncertain optimization problem from infeasibility.
- Classical approach: define a support set for the uncertainties, and use duality arguments to construct the robust counterpart (RC).
- RC: a deterministic equivalent of the uncertain optimization problem that ensures feasibility with probability 1.
- E.g. Soyster (1973), Ben-Tal and Nemirovski (1998), Bertsimas and Sim (2004), El Ghaoui, Oustry, and Lebret (1998).

Affinely Adjustable Robust Counterpart (AARC)

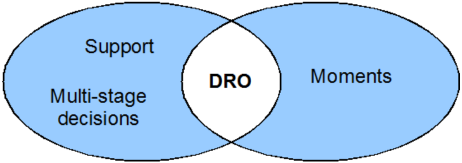
- Ben-Tal, Goryashko, Guslitzer, and Nemirovski (2004) introduced Affinely-Adjustable RC (AARC), to allow delayed decision-making. (aka LDR)
- Decisions are affine functions of uncertainties.
- Can model two-stage and multi-stage problems.
- Advantage: Tractable approximation of many stochastic programs, and is distribution-free.
- Disadvantage: Can be too conservative for some problems.

Using Moments for RO

- A different perspective on RO: *Moment* Information
- Goes back to Scarf (1958) Distribution-free newsvendor with known mean and variance.
- El Ghaoui, Oks, and Oustry (2004): Use mean and covariance to compute WVaR for portfolio of assets.
- Popescu (2007) Use mean-covariance information to robustly optimize expected utility.

DRO Combines These Ideas

- Distributionally Robust Optimization (DRO) is at the intersection of these research threads.



- Existing work:
  - Chen, Sim, Sun, and Zhang (2008) LDR extensions
  - Chen and Sim (2009) Goal driven optimization
  - Natarajan, Sim, and Uichanco (2009) Expected utility

Design Goals of DRO

- To design a unified theoretical framework for DRO
- To build a software modeling tool to solve DRO problems within this framework
- Key ingredients:
    - Multi-stage decisions, but more flexible compared to LDRs.
    - Distribution-free, but partially characterized by support, moments, and directional deviations (Chen, Sim, and Sun, 2007).

Outline

Notation

- $\tilde{\boldsymbol{z}} \in \Re^N$: uncertainties with distribution $\mathbb{P} \in \mathbb{F}$.
- $\boldsymbol{x} \in \Re^n$: "here-and-now" decision variables.
- $\boldsymbol{y}^k(\tilde{\boldsymbol{z}}) \in \Re^{m_k}$: "wait-and-see" decision rules, $\forall k \in [K]$.
- Each decision rule need not depend on the full uncertainty vector. We introduce K information index sets, to capture the dependency structure, denoted by $I_k \subseteq [N], \forall k \in [K]$.

$$\boldsymbol{y}^k \in \mathcal{Y}(m_k, N, I_k) \quad \forall k \in [K]$$

$$\mathcal{Y}(m, N, I) \triangleq \left\{ \boldsymbol{f} : \Re^N \rightarrow \Re^m \ : \ \boldsymbol{f}\left(\boldsymbol{z} + \sum_{i \notin I} \lambda_i \boldsymbol{e}^i \right) = \boldsymbol{f}(\boldsymbol{z}), \forall \boldsymbol{\lambda} \in \Re^N \right\}$$

$$I \subseteq [N] \equiv \{1, 2, \ldots, N\}$$

Information Index Set Example

- Suppose you have a Decision Rule, $\boldsymbol{y}^1(\tilde{\boldsymbol{z}})$ which has information index set I_1, i.e. $\boldsymbol{y}^1 \in \mathcal{Y}(m, N, I_1)$.
- The information index set contains two elements, $I_1 = \{2, 3\}$.
- Then, the Decision Rule can be equivalently expressed as $\boldsymbol{y}^1(\tilde{\boldsymbol{z}}) = \boldsymbol{y}^1(\tilde{z}_2, \tilde{z}_3)$.

Information Index Sets

- Allows us to handle more general non-anticipative scenarios than multi-stage problems.
- For a multistage problem with sequential decisions, we have

$$I_1 \subseteq I_2 \subseteq \ldots \subseteq I_K$$

- Reflects the requirement that we only can make use of uncertainties *in the past*.
- Other set inclusions (especially occuring in network problems) lead to more general non-anticipative constraints.

Model of Uncertainty

- Actual uncertainty distribution $\mathbb{P}$ lies in a family of distributions $\mathbb{F}$
- Partially characterized by distributional properties:
  - Support $\mathcal{W}$: Tractable conic representable, full-dim.
  - Mean Support $\hat{\mathcal{W}}$: Tractable conic representable.
  - Covariance matrix $\boldsymbol{\Sigma}$: Positive definite
  - Upper bounds on Directional Deviations for stochastically independent components $(\boldsymbol{H}_\sigma, \boldsymbol{\sigma_f}, \boldsymbol{\sigma_b})$.
    - The original uncertainties may not have independent components, but we allow the possibility that a linear transformation of the uncertainties by $\boldsymbol{H}_\sigma$ may have independent components.
    - i.e. $\tilde{\boldsymbol{w}} = \boldsymbol{H}_\sigma \tilde{\boldsymbol{z}}$ might have independent components.

General Model

$$
\begin{array}{lll}
{Z_{GEN}}^* = & & \\
\min\limits_{\boldsymbol{x},\{\boldsymbol{y}^k(\cdot)\}_{k=1}^K} & \boldsymbol{c}^{0\prime}\boldsymbol{x} + \sup\limits_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\sum\limits_{k=1}^K \boldsymbol{d}^{0,k\prime}\boldsymbol{y}^k(\tilde{\boldsymbol{z}})\right) & \\
\text{s.t.} & \boldsymbol{c}^{l\prime}\boldsymbol{x} + \sup\limits_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\sum\limits_{k=1}^K \boldsymbol{d}^{l,k\prime}\boldsymbol{y}^k(\tilde{\boldsymbol{z}})\right) \le b_l & \forall l \in [M] \\
& \boldsymbol{T}(\tilde{\boldsymbol{z}})\boldsymbol{x} + \sum\limits_{k=1}^K \boldsymbol{U}^k\boldsymbol{y}^k(\tilde{\boldsymbol{z}}) = \boldsymbol{v}(\tilde{\boldsymbol{z}}) & \\
& \underline{\boldsymbol{y}}^k \le \boldsymbol{y}^k(\tilde{\boldsymbol{z}}) \le \overline{\boldsymbol{y}}^k & \forall k \in [K] \\
& \boldsymbol{x} \ge \boldsymbol{0} & \\
& \boldsymbol{y}^k \in \mathcal{Y}(m_k, N, I_k) & \forall k \in [K]
\end{array}
$$

- Convention: All (in)equalities involving uncertainties are taken to hold with probability 1 over all distributions $\mathbb{P} \in \mathbb{F}$.

General Model (data in red)

$$
\begin{aligned}
&{Z_{GEN}}^* = \\
&\min_{\boldsymbol{x},\{\boldsymbol{y}^k(\cdot)\}_{k=1}^K} \quad \boldsymbol{c}^{0\prime}\boldsymbol{x} + \sup_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\sum_{k=1}^K \boldsymbol{d}^{0,k\prime}\boldsymbol{y}^k(\tilde{\boldsymbol{z}})\right) \\
&\text{s.t.} \quad \boldsymbol{c}^{l\prime}\boldsymbol{x} + \sup_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\sum_{k=1}^K \boldsymbol{d}^{l,k\prime}\boldsymbol{y}^k(\tilde{\boldsymbol{z}})\right) \leq b_l \quad \forall l \in [M] \\
&\qquad \boldsymbol{T}(\tilde{\boldsymbol{z}})\boldsymbol{x} + \sum_{k=1}^K \boldsymbol{U}^k \boldsymbol{y}^k(\tilde{\boldsymbol{z}}) = \boldsymbol{v}(\tilde{\boldsymbol{z}}) \\
&\qquad \underline{\boldsymbol{y}}^k \leq \boldsymbol{y}^k(\tilde{\boldsymbol{z}}) \leq \overline{\boldsymbol{y}}^k \quad \forall k \in [K] \\
&\qquad \boldsymbol{x} \geq \boldsymbol{0} \\
&\qquad \boldsymbol{y}^k \in \mathcal{Y}(m_k, N, I_k) \quad \forall k \in [K]
\end{aligned}
$$

- We assume $\boldsymbol{T}(\tilde{\boldsymbol{z}}), \boldsymbol{v}(\tilde{\boldsymbol{z}})$, are affine functions of $\tilde{\boldsymbol{z}}$, i.e. $\boldsymbol{T}(\tilde{\boldsymbol{z}}) = \boldsymbol{T}^0 + \sum_{i=1}^N \boldsymbol{T}^i \tilde{z}_i$, $\boldsymbol{v}(\tilde{\boldsymbol{z}}) = \boldsymbol{v}^0 + \sum_{i=1}^N \boldsymbol{v}^i \tilde{z}_i$

Example: Piecewise-linear terms

- Common to find $\mathrm{E}_{\mathbb{P}}\left((\cdot)^{+}\right)$ in terms in optimization problems in the OM setting. E.g. Newsvendor / inventory models.
- A distributionally robust version can be cast in our framework too:

$$\begin{array}{l} \sup_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\boldsymbol{y}(\tilde{\boldsymbol{z}})^{+}\right) \leq \boldsymbol{b} \\ \boldsymbol{y} \in \mathcal{Y}(m, N, I) \end{array}$$

$$\Updownarrow$$

$$\begin{array}{rcl} \sup_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\boldsymbol{s}(\tilde{\boldsymbol{z}})\right) & \leq & \boldsymbol{b} \\ \boldsymbol{s}(\tilde{\boldsymbol{z}}) & \geq & \boldsymbol{0} \\ \boldsymbol{s}(\tilde{\boldsymbol{z}}) & \geq & \boldsymbol{y}(\tilde{\boldsymbol{z}}) \\ \boldsymbol{s}, \boldsymbol{y} & \in & \mathcal{Y}(m, N, I) \end{array}$$

Using Linear Decision Rules (LDRs)

- General model is intractable. Intuitively, the space of all functions is "too big".
- Following Ben-Tal et. al. (2004), we restrict ourselves to LDRs, where the recourse decisions are affinely dependent on the model uncertainties.
- Define the space of LDRs

$$\mathcal{L}(m,N,I) = \left\{ \boldsymbol{f} : \Re^N \to \Re^m : \exists \left(\boldsymbol{y}^0, \boldsymbol{Y}\right) \in \Re^m \times \Re^{m\times N} : \begin{array}{l} \boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{y}^0 + \boldsymbol{Y}\boldsymbol{z} \\ \boldsymbol{Y}\boldsymbol{e}^i = \boldsymbol{0}, \forall i \notin I \end{array} \right.$$

- Use this approximate $\mathcal{Y}(m,N,I)$

LDR Approximation of General Model

$$
\begin{array}{lll}
Z_{LDR} = & & \\
\min\limits_{\boldsymbol{x},\{\boldsymbol{y}^k(\cdot)\}_{k=1}^K} & \boldsymbol{c}^{0\prime}\boldsymbol{x} + \sup\limits_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\sum_{k=1}^K \boldsymbol{d}^{0,k\prime}\boldsymbol{y}^k(\tilde{\boldsymbol{z}})\right) & \\
\text{s.t.} & \boldsymbol{c}^{l\prime}\boldsymbol{x} + \sup\limits_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\sum_{k=1}^K \boldsymbol{d}^{l,k\prime}\boldsymbol{y}^k(\tilde{\boldsymbol{z}})\right) \leq b_l & \forall l \in [M] \\
& \boldsymbol{T}(\tilde{\boldsymbol{z}})\boldsymbol{x} + \sum_{k=1}^K \boldsymbol{U}^k\boldsymbol{y}^k(\tilde{\boldsymbol{z}}) = \boldsymbol{v}(\tilde{\boldsymbol{z}}) & \\
& \underline{\boldsymbol{y}}^k \leq \boldsymbol{y}^k(\tilde{\boldsymbol{z}}) \leq \overline{\boldsymbol{y}}^k & \forall k \in [K] \\
& \boldsymbol{x} \geq \boldsymbol{0} & \\
& \boldsymbol{y}^k \in \mathcal{L}(m_k, N, I_k) & \forall k \in [K]
\end{array}
$$

LDR Approx. of General Model (Explicit)

$$
\begin{array}{lll}
Z_{LDR} = & & \\
\min\limits_{\boldsymbol{x},\{\boldsymbol{y}^{0,k},\boldsymbol{Y}^k\}_{k=1}^K} & \boldsymbol{c}^{0\prime}\boldsymbol{x} + \sum\limits_{k=1}^K \boldsymbol{d}^{0,k\prime}\boldsymbol{y}^{0,k} + \sup\limits_{\hat{\boldsymbol{z}}\in\hat{\mathcal{W}}} \left(\sum\limits_{k=1}^K \boldsymbol{d}^{0,k\prime}\boldsymbol{Y}^k\hat{\boldsymbol{z}}\right) & \\
\text{s.t.} & \boldsymbol{c}^{l\prime}\boldsymbol{x} + \sum\limits_{k=1}^K \boldsymbol{d}^{l,k\prime}\boldsymbol{y}^{0,k} + \sup\limits_{\hat{\boldsymbol{z}}\in\hat{\mathcal{W}}} \left(\sum\limits_{k=1}^K \boldsymbol{d}^{l,k\prime}\boldsymbol{Y}^k\hat{\boldsymbol{z}}\right) \le b_l & \forall l \in [M] \\
& \boldsymbol{T}^0\boldsymbol{x} + \sum\limits_{k=1}^K \boldsymbol{U}^k\boldsymbol{y}^{0,k} = \boldsymbol{v}^0 & \\
& \boldsymbol{T}^j\boldsymbol{x} + \sum\limits_{k=1}^K \boldsymbol{U}^k\boldsymbol{Y}^k\boldsymbol{e}^j = \boldsymbol{v}^j & \forall j \in [N] \\
& \underline{\boldsymbol{y}}^k \le \boldsymbol{y}^{0,k} + \boldsymbol{Y}^k\boldsymbol{z} \le \overline{\boldsymbol{y}}^k \qquad \forall \boldsymbol{z}\in\mathcal{W} & \forall k \in [K] \\
& \boldsymbol{Y}^k\boldsymbol{e}^j = \boldsymbol{0} & \forall j \notin I_k, \forall k \in \\
& \boldsymbol{x} \ge \boldsymbol{0} &
\end{array}
$$

Is LDR Approximation too Conservative?

- LDR approximation of General Model is tractable, but is it too conservative?
- will proceed by using the basic LDR approximation as a starting point, and aim to find other more complex decision rules which are better.

$$Z_{GEN}{}^{*} \leq ??? \leq ??? \leq Z_{LDR}$$

Outline

Segregating the Uncertainties

- Idea: to re-map the uncertainties into a higher-dimensional space, and apply an LDR on the new uncertainties, for more flexibility.
- E.g. for a scalar uncertainty, we can segregate into positive and negative parts (as was done by Chen, Sim, Sun, and Zhang, 2008).

$$\tilde{z} = \underbrace{\tilde{z}^+}_{\tilde{\zeta}_1} - \underbrace{\tilde{z}^-}_{\tilde{\zeta}_2}$$

- Can segregate more generally into distinct segments of the real line.

General Segregations

- Consider $L+1$ *partition points* on the extended real line
- Ordered, denoted by $\{\xi_k\}_{k=1}^{L+1}$, $\xi_1 = -\infty, \xi_{L+1} = +\infty$.
- Segregating mapping function:

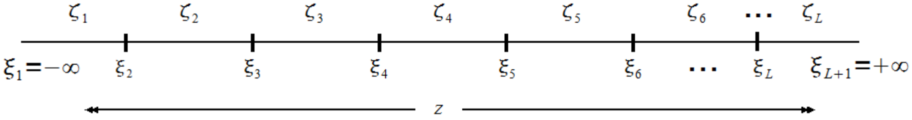
$$\boldsymbol{M} : \Re \to \Re^L, \boldsymbol{\zeta} = \boldsymbol{M}(z)$$

$$\zeta_k = \begin{cases} z & \text{if } \xi_k \leq z \leq \xi_{k+1} \\ \xi_k & \text{if } z \leq \xi_k \\ \xi_{k+1} & \text{if } z \geq \xi_{k+1} \end{cases}$$

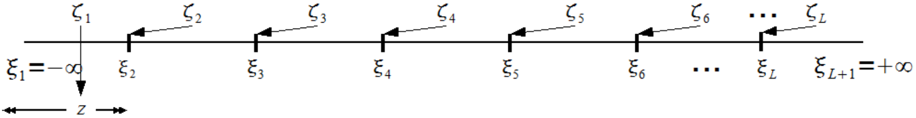
- Define image of supports under the mapping:

$$\begin{aligned} \mathcal{V}^* &\triangleq \{\boldsymbol{M}(\boldsymbol{z}) : \boldsymbol{z} \in \mathcal{W}\} \\ \hat{\mathcal{V}}^* &\triangleq \left\{\boldsymbol{M}(\hat{\boldsymbol{z}}) : \hat{\boldsymbol{z}} \in \hat{\mathcal{W}}\right\} \end{aligned}$$

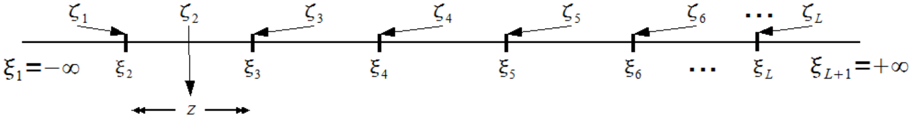
General Segregations (Pictorial)



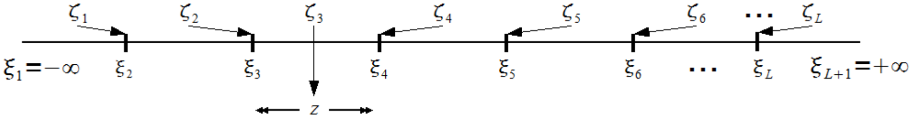
General Segregations (Pictorial)



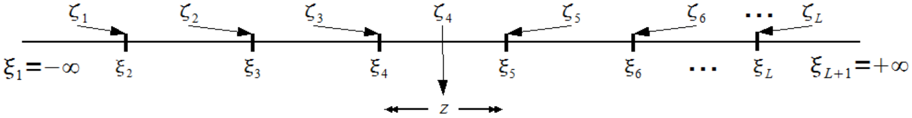
General Segregations (Pictorial)



General Segregations (Pictorial)



General Segregations (Pictorial)



Segregations are Affinely-infertible

$$\sum_{i=1}^{L} \zeta_i = z + \sum_{i=2}^{L} \xi_i \iff z = \sum_{i=1}^{L} \zeta_i - \sum_{i=2}^{L} \xi_i$$

- In general, for vector uncertainties, we have:

$$\begin{array}{rcl} \boldsymbol{z} & = & \boldsymbol{F}\boldsymbol{M}(\boldsymbol{z}) + \boldsymbol{g} \quad \forall \boldsymbol{z} \in \Re^N \\ \boldsymbol{F} & = & \left[\begin{array}{cccc} \boldsymbol{I} & \boldsymbol{I} & \ldots & \boldsymbol{I} \end{array} \right] \\ \boldsymbol{g} & = & \text{constant} \end{array}$$

- We can now define a new set of LDRs on the (higher-dimensional) segregated uncertainties, i.e.

$$\boldsymbol{y}^k(\tilde{\boldsymbol{z}}) = \boldsymbol{r}^k(\boldsymbol{M}(\tilde{\boldsymbol{z}})) = \boldsymbol{r}^{0,k} + \boldsymbol{R}^k \boldsymbol{M}(\tilde{\boldsymbol{z}}), \ \forall k \in [K]$$

General Model under SLDR (Exact)

$$Z_{SLDR}{}^{*} =$$

$$\begin{array}{lll} \min\limits_{\boldsymbol{x},\{\boldsymbol{r}^k(\cdot)\}_{k=1}^K} & \boldsymbol{c}^{0\prime}\boldsymbol{x} + \sum\limits_{k=1}^K \boldsymbol{d}^{0,k\prime}\boldsymbol{r}^{0,k} + \sup\limits_{\hat{\boldsymbol{\zeta}}\in\hat{\mathcal{V}}^*}\left(\sum\limits_{k=1}^K \boldsymbol{d}^{0,k\prime}\boldsymbol{R}^k\hat{\boldsymbol{\zeta}}\right) & \\ \text{s.t.} & \boldsymbol{c}^{l\prime}\boldsymbol{x} + \sum\limits_{k=1}^K \boldsymbol{d}^{l,k\prime}\boldsymbol{r}^{0,k} + \sup\limits_{\hat{\boldsymbol{\zeta}}\in\hat{\mathcal{V}}^*}\left(\sum\limits_{k=1}^K \boldsymbol{d}^{l,k\prime}\boldsymbol{R}^k\hat{\boldsymbol{\zeta}}\right) \le b_l & \forall l \in [M] \\ & \boldsymbol{T}(\boldsymbol{F}\boldsymbol{\zeta}+\boldsymbol{g})\boldsymbol{x} + \left(\sum\limits_{k=1}^K \boldsymbol{U}^k\boldsymbol{r}^{0,k} + \boldsymbol{U}^k\boldsymbol{R}^k\boldsymbol{\zeta}\right) = \boldsymbol{v}(\boldsymbol{F}\boldsymbol{\zeta}+\boldsymbol{g}) & \forall \boldsymbol{\zeta} \in \mathcal{V}^* \\ & \underline{\boldsymbol{y}}^k \le \boldsymbol{r}^{0,k} + \boldsymbol{R}^k\boldsymbol{\zeta} \le \overline{\boldsymbol{y}}^k & \forall \boldsymbol{\zeta} \in \mathcal{V}^*, \forall k \in [K] \\ & \boldsymbol{x} \ge \boldsymbol{0} & \\ & \boldsymbol{r}^k \circ \boldsymbol{M} \in \mathcal{Y}(m_k, N, I_k) & \forall k \in [K] \end{array}$$

General Model Under SLDR (Approximated)

$$
\begin{aligned}
Z_{SLDR} = & \\
\min_{\boldsymbol{x}, \{\boldsymbol{r}^{0,k}, \boldsymbol{R}^k\}_{k=1}^K} \quad & \boldsymbol{c}^{0\prime}\boldsymbol{x} + \sum_{k=1}^K \boldsymbol{d}^{0,k\prime}\boldsymbol{r}^{0,k} + \sup_{\hat{\zeta}\in\hat{\mathcal{V}}}\left(\sum_{k=1}^K \boldsymbol{d}^{0,k\prime}\boldsymbol{R}^k\hat{\zeta}\right) \\
\text{s.t.} \quad & \boldsymbol{c}^{l\prime}\boldsymbol{x} + \sum_{k=1}^K \boldsymbol{d}^{l,k\prime}\boldsymbol{r}^{0,k} + \sup_{\hat{\zeta}\in\hat{\mathcal{V}}}\left(\sum_{k=1}^K \boldsymbol{d}^{l,k\prime}\boldsymbol{R}^k\hat{\zeta}\right) \le b_l \quad \forall l \in [M] \\
& \boldsymbol{\mathcal{T}}^0\boldsymbol{x} + \sum_{k=1}^K \boldsymbol{U}^k\boldsymbol{r}^{0,k} = \boldsymbol{\nu}^0 \\
& \boldsymbol{\mathcal{T}}^j\boldsymbol{x} + \sum_{k=1}^K \boldsymbol{U}^k\boldsymbol{R}^k\boldsymbol{e}^j = \boldsymbol{\nu}^j \qquad \forall j \in [N_E] \\
& \underline{\boldsymbol{y}}^k \le \boldsymbol{r}^{0,k} + \boldsymbol{R}^k\zeta \le \overline{\boldsymbol{y}}^k \qquad \forall \zeta \in \mathcal{V} \quad \forall k \in [K] \\
& \boldsymbol{R}^k\boldsymbol{e}^j = \boldsymbol{0} \qquad \forall j \notin \Phi_k, \forall k \in [K] \\
& \boldsymbol{x} \ge \boldsymbol{0}
\end{aligned}
$$

Properties of SLDR

Theorem

For the general problem, $Z_{SLDR} \leq Z_{LDR}$

- Increases the size of the problem (and computational effort needed), but retains the linear structure.
- Can we do better?

$$Z_{GEN}{}^{*} \leq ??? \leq Z_{SLDR} \leq Z_{LDR}$$

- Will base further improvement on the SLDR.

Outline

Robust Bounds on $\sup_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\left(r^0 + \boldsymbol{r}'\tilde{\boldsymbol{\zeta}}\right)^+\right)$

- We will need this later for the BDLDR
- Different pairs of distributional properties lead to different upper bounds.
    - Mean + Support
    - Mean + Covariance
    - Mean + Directional Deviations
- Innermost argument: Scalar-valued SLDR

(Mean + Support) & (Mean + Covariance)

- To bound $\sup_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\left(r^0 + \boldsymbol{r}'\tilde{\boldsymbol{\zeta}}\right)^+\right)$
- $\mathbb{F}$ characterized by $\hat{\mathcal{V}}$ and $\mathcal{V}$ (Mean, Support)

$$\pi^1\left(r^0, \boldsymbol{r}\right) \triangleq \inf_{\boldsymbol{s}\in\Re^{N_E}} \left(\sup_{\hat{\boldsymbol{\zeta}}\in\hat{\mathcal{V}}}\left\{\boldsymbol{s}'\hat{\boldsymbol{\zeta}}\right\} + \sup_{\boldsymbol{\zeta}\in\mathcal{V}}\left(\max\left\{r^0 + \boldsymbol{r}'\boldsymbol{\zeta} - \boldsymbol{s}'\boldsymbol{\zeta}, -\boldsymbol{s}'\boldsymbol{\zeta}\right\}\right)\right)$$

- $\mathbb{F}$ characterized by $\hat{\mathcal{V}}$ and $\boldsymbol{\Sigma}$ (Mean, Covariance)

$$\pi^2\left(r^0, \boldsymbol{r}\right) \triangleq \inf_{\boldsymbol{y}\in\{\boldsymbol{y}:\boldsymbol{F}'\boldsymbol{y}=\boldsymbol{r}\}} \left\{\sup_{\hat{\boldsymbol{\zeta}}\in\hat{\mathcal{V}}}\left\{\frac{1}{2}\left(r^0 + \boldsymbol{r}'\hat{\boldsymbol{\zeta}}\right) + \frac{1}{2}\sqrt{(r^0 + \boldsymbol{r}'\hat{\boldsymbol{\zeta}})^2 + \boldsymbol{y}'\boldsymbol{\Sigma}\boldsymbol{y}}\right\}\right\}$$

Mean + Directional Deviation

- $\mathbb{F}$ characterized by $\hat{\mathcal{V}}$ and $(\boldsymbol{H}_\sigma, \boldsymbol{\sigma_f}, \boldsymbol{\sigma_b})$ (Mean, Directional Devations)

$$\pi^3\left(r^0, \boldsymbol{r}\right) \triangleq \inf_{\substack{s^0, \boldsymbol{s}, x^0, \boldsymbol{x} \\ x^0 + \boldsymbol{x}'\boldsymbol{g}_\sigma = r^0 \\ \boldsymbol{F}_\sigma' \boldsymbol{x} = \boldsymbol{r}}} \left\{ \begin{array}{lll} (r^0 - s^0 - \boldsymbol{s}'\boldsymbol{g}_\sigma) & + & \sup_{\hat{\boldsymbol{\zeta}} \in \hat{\mathcal{V}}} (\boldsymbol{r}' - \boldsymbol{s}'\boldsymbol{F}_\sigma)\hat{\boldsymbol{\zeta}} \\ & + & \psi(s^0 - x^0, \boldsymbol{s} - \boldsymbol{x}) + \psi(s^0, \boldsymbol{s}) \end{array} \right.$$

- Where $\psi(x^0, \boldsymbol{x}) = \inf_{\lambda > 0} \left\{ \frac{\lambda}{e} \exp\left(\frac{1}{\lambda} \sup_{\hat{\boldsymbol{z}}_\sigma \in \hat{\mathcal{W}}_\sigma} \left\{ x^0 + \boldsymbol{x}'\hat{\boldsymbol{z}}_\sigma \right\} + \frac{\|\boldsymbol{u}\|_2^2}{2\lambda^2} \right) \right\}$

$$\begin{aligned} u_j &= \max\left\{ x_j \sigma_{f,j}, -x_j \sigma_{b,j} \right\} \\ \boldsymbol{F}_\sigma &= \boldsymbol{H}_\sigma \boldsymbol{F} \\ \boldsymbol{g}_\sigma &= \boldsymbol{H}_\sigma \boldsymbol{g} \end{aligned}$$

- $\mathbb{F}$ characterized by all of these properties
  - Use infimal convolution to combine bounds
  - Resulting bound is better than individual bounds
  - But not tight

$$\pi\left(r^0, \boldsymbol{r}\right) = \begin{array}{ll} \min & \pi^1\left(r^{0,1}, \boldsymbol{r}^1\right) + \pi^2\left(r^{0,2}, \boldsymbol{r}^2\right) + \pi^3\left(r^{0,3}, \boldsymbol{r}^3\right) \\ \text{s.t.} & r^0 = r^{0,1} + r^{0,2} + r^{0,3} \\ & \boldsymbol{r} = \boldsymbol{r}^1 + \boldsymbol{r}^2 + \boldsymbol{r}^3 \end{array}$$

Outline

Introduction

DRO Framework

Segregated LDR

Robust bounds on $\sup_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left((\cdot)^{+}\right)$

Bi-Deflected LDR

Conclusion

Review of Deflected LDR (DLDR)

- Originally introduced by Chen, Sim, Sun, and Zhang (2008) to circumvent restrictiveness imposed by the LDR.
    - The structure of the set of constraints might allow some piecewise-linear decision rules to be used instead.
    - Apply bounds on expected positive part of an LDR to exploit piecewise-linearity.
- Bi-deflected LDR seeks to improve and extend the DLDR.
    - Improve: Reduce the objective even further.
    - Extend: Explore modeling scenarios (non-anticipativity, expectations in constraints) which the DLDR didn't handle.

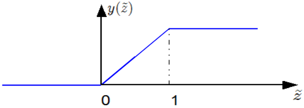
Motivating Example

- Scalar uncertainty $\tilde{z}$ with unknown distribution $\mathbb{P}$ in a family $\mathbb{F}$.
- $\mathbb{F}$ defined by infinite support, zero mean, and unit variance. Consider the following problem:

$$\begin{array}{ll} \min\limits_{y(\cdot)} & \sup\limits_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(|y(\tilde{z}) - \tilde{z}|\right) \\ & 0 \leq y(\tilde{z}) \leq 1 \\ & y \in \mathcal{Y}\left(1, 1, \{1\}\right) \end{array}$$

- Solution is pretty straightforward: piecewise-linear recourse function:

$$y_{sol}(\tilde{z}) = \begin{cases} \tilde{z} & \text{if } 0 \leq \tilde{z} \leq 1 \\ 0 & \text{if } \tilde{z} \leq 0 \\ 1 & \text{if } \tilde{z} \geq 1 \end{cases}$$



Example Reformulated into DRO Framework

- Apply the identities $x = x^+ - x^-, |x| = x^+ + x^-$.
- Model reformulates into DRO framework:

$$
\begin{array}{ll}
\min\limits_{y(\cdot)} & \sup\limits_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(|y(\tilde{z}) - \tilde{z}|\right) \\
 & 0 \leq y(\tilde{z}) \leq 1 \\
 & y \in \mathcal{Y}\left(1, 1, \{1\}\right)
\end{array}
$$

$$\Updownarrow$$

$$
\begin{array}{ll}
\min\limits_{y(\cdot),u(\cdot),v(\cdot)} & \sup\limits_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(u(\tilde{z}) + v(\tilde{z})\right) \\
\text{s.t.} & u(\tilde{z}) - v(\tilde{z}) = y(\tilde{z}) - \tilde{z} \\
 & 0 \leq y(\tilde{z}) \leq 1 \\
 & u(\tilde{z}), v(\tilde{z}) \geq 0 \\
 & y, u, v \in \mathcal{Y}\left(1, 1, \{1\}\right)
\end{array}
$$

Using Different Decision Rules

- Using LDR, problem is infeasible, objective $= +\infty$.
- Using DLDR (of Chen et. al. 2008), we get the piecewise-linear decision rule:

$$\begin{aligned} \hat{u}_D(z) &= \left(u^0 + uz\right)^+ + \left(v^0 + vz\right)^- \\ \hat{v}_D(z) &= \left(v^0 + vz\right)^+ + \left(u^0 + uz\right)^- \\ y(z) &= y^0 + yz \end{aligned}$$

- After applying bounds, becomes an SOCP

$$\begin{aligned} Z_{DLDR} = \min_{u^0,u,v^0,v} \quad & \left\| \begin{pmatrix} u^0 \\ u \end{pmatrix} \right\|_2 + \left\| \begin{pmatrix} v^0 \\ v \end{pmatrix} \right\|_2 \\ \text{s.t.} \quad & u - v = -1 \\ & 0 \le u^0 - v^0 \le 1 \end{aligned}$$

- Objective $= 1$.

Different Decision Rules (II)

- Consider this hypothetical decision rule, which satisfies the model constraints:

$$
\begin{aligned}
\hat{u}(z) &= \left(u^0 + uz\right)^+ + \left(v^0 + vz\right)^- + \left(y^0 + yz\right)^- \\
\hat{v}(z) &= \left(v^0 + vz\right)^+ + \left(u^0 + uz\right)^- + \left(y^0 - 1 + yz\right)^+ \\
\hat{y}(z) &= \left(y^0 + yz\right)^+ - \left(y^0 - 1 + yz\right)^+
\end{aligned}
$$

- Applying robust bounds, problem transforms into the SOCP:

$$
\begin{array}{rl}
\min\limits_{y^0, y, u^0, u, v^0, v} & \left\| \begin{pmatrix} u^0 \\ u \end{pmatrix} \right\|_2 + \left\| \begin{pmatrix} v^0 \\ v \end{pmatrix} \right\|_2 + \frac{1}{2} \left\| \begin{pmatrix} y^0 \\ y \end{pmatrix} \right\|_2 + \frac{1}{2} \left\| \begin{pmatrix} y^0 - 1 \\ y \end{pmatrix} \right\|_2 - \frac{1}{2} \\
\text{s.t.} & u^0 - v^0 = y^0 \\
& u - v = y - 1
\end{array}
$$

- Objective = 0.707 (better!)
- Aim to find algorithm to automatically predict this decision rule

Two-stage Bi-deflected LDR Problem Setup

- Consider a simpler two-stage model first:

$$\begin{array}{rl}\min\limits_{\boldsymbol{x},\boldsymbol{r}(\cdot)} & \boldsymbol{c}'\boldsymbol{x} + \sup\limits_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\boldsymbol{d}'\boldsymbol{r}(\tilde{\zeta})\right) \\ \text{s.t.} & \boldsymbol{\mathcal{T}}(\tilde{\zeta})\boldsymbol{x} + \boldsymbol{U}\boldsymbol{r}(\tilde{\zeta}) = \boldsymbol{\nu}(\tilde{\zeta}) \\ & \underline{\boldsymbol{y}} \leq \boldsymbol{r}(\tilde{\zeta}) \leq \overline{\boldsymbol{y}} \\ & \boldsymbol{r} \in \mathcal{L}(m, N_E, [N_E])\end{array}$$

- Define the index sets of non-infinite bounds:

$$\underline{J} = \left\{i \in [m] : \underline{y}_i > -\infty\right\}$$
$$\overline{J} = \{i \in [m] : \overline{y}_i < +\infty\}$$

Two-stage BDLDR Sub-problems

- We solve a series of sub-problems:

$$\underline{\forall i \in \underline{J}}$$

$$\begin{aligned} \min_{\boldsymbol{p}} \quad & \boldsymbol{d}'\boldsymbol{p} \\ \text{s.t.} \quad & \boldsymbol{U}\boldsymbol{p} = \boldsymbol{0} \\ & p_i = 1 \\ & p_j \geq 0 \qquad \forall j \in \underline{J} \\ & p_j \leq 0 \qquad \forall j \in \overline{J} \setminus \{i\} \end{aligned}$$

Feasible $i \in \underline{J}^\circ$

$$\underline{\forall i \in \overline{J}}$$

$$\begin{aligned} \min_{\boldsymbol{q}} \quad & \boldsymbol{d}'\boldsymbol{q} \\ \text{s.t.} \quad & \boldsymbol{U}\boldsymbol{q} = \boldsymbol{0} \\ & q_i = -1 \\ & q_j \leq 0 \qquad \forall j \in \overline{J} \\ & q_j \geq 0 \qquad \forall j \in \underline{J} \setminus \{i\} \end{aligned}$$

Feasible $i \in \overline{J}^\circ$

Two-stage BDLDR

- Consider an LDR which satisfies the relaxed constraints:

$$\begin{aligned} \boldsymbol{\mathcal{T}}(\tilde{\boldsymbol{\zeta}})\boldsymbol{x} + \boldsymbol{U}\boldsymbol{r}(\tilde{\boldsymbol{\zeta}}) &= \boldsymbol{\nu}(\tilde{\boldsymbol{\zeta}}) & \\ r_j(\tilde{\boldsymbol{\zeta}}) &\geq \underline{y}_j & \forall j \in \underline{J} \setminus \underline{J}^\circ \\ r_j(\tilde{\boldsymbol{\zeta}}) &\leq \overline{y}_j & \forall j \in \overline{J} \setminus \overline{J}^\circ \end{aligned}$$

- The two-stage BDLDR is defined as:

$$\hat{\boldsymbol{r}}(\tilde{\boldsymbol{\zeta}}) \triangleq \underbrace{\boldsymbol{r}(\tilde{\boldsymbol{\zeta}})}_{\text{Linear Part}} + \sum_{i \in \underline{J}^\circ} \left(r_i(\tilde{\boldsymbol{\zeta}}) - \underline{y}_i \right)^- \underbrace{\bar{\boldsymbol{p}}^i} + \sum_{i \in \overline{J}^\circ} \left(r_i(\tilde{\boldsymbol{\zeta}}) - \overline{y}_i \right)^+ \underbrace{\bar{\boldsymbol{q}}^i}$$

Optimal solution of the i^{th} sub-problem (Deflected Part)

Two-stage BDLDR Properties

Theorem

The Bi-Deflected Linear Decision Rule, $\hat{\boldsymbol{r}}(\tilde{\boldsymbol{\zeta}})$, satisfies the following properties

1. $\boldsymbol{U}\hat{\boldsymbol{r}}(\tilde{\boldsymbol{\zeta}}) = \boldsymbol{U}\boldsymbol{r}(\tilde{\boldsymbol{\zeta}})$
2. $\underline{\boldsymbol{y}} \le \hat{\boldsymbol{r}}(\tilde{\boldsymbol{\zeta}}) \le \overline{\boldsymbol{y}}$

- For Reference:

$$\hat{\boldsymbol{r}}(\tilde{\boldsymbol{\zeta}}) \triangleq \underbrace{\boldsymbol{r}(\tilde{\boldsymbol{\zeta}})}_{\text{Linear Part}} + \sum_{i \in \underline{J}^\circ} \left(r_i(\tilde{\boldsymbol{\zeta}}) - \underline{y}_i \right)^- \underbrace{\bar{\boldsymbol{p}}^i} + \sum_{i \in \overline{J}^\circ} \left(r_i(\tilde{\boldsymbol{\zeta}}) - \overline{y}_i \right)^+ \underbrace{\bar{\boldsymbol{q}}^i}$$

Optimal solution of the i^{th} sub-problem (Deflected Part)

BDLDR Approximation of Two-stage Problem

- Using the BDLDR, the problem becomes:

$$
\begin{array}{rl}
Z_{BDLDR} = & \\
\min\limits_{\boldsymbol{x}, \boldsymbol{r}(\cdot)} & \boldsymbol{c}'\boldsymbol{x} + \sup\limits_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}} \left(\boldsymbol{d}'\boldsymbol{r}(\tilde{\boldsymbol{\zeta}}) \right) \\
& + \sum\limits_{i \in \underline{J}_R^\circ} \sup\limits_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}} \left(\left(r_i(\tilde{\boldsymbol{\zeta}}) - \underline{y}_i \right)^- \right) \boldsymbol{d}'\bar{\boldsymbol{p}}^i \\
& + \sum\limits_{i \in \overline{J}_R^\circ} \sup\limits_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}} \left(\left(r_i(\tilde{\boldsymbol{\zeta}}) - \overline{y}_i \right)^+ \right) \boldsymbol{d}'\bar{\boldsymbol{q}}^i \\
\text{s.t.} & \\
& \boldsymbol{\mathcal{T}}(\tilde{\boldsymbol{\zeta}})\boldsymbol{x} + \boldsymbol{U}\boldsymbol{r}(\tilde{\boldsymbol{\zeta}}) = \boldsymbol{\nu}(\tilde{\boldsymbol{\zeta}}) \\
& r_j(\tilde{\boldsymbol{\zeta}}) \geq \underline{y}_j \qquad \forall j \in \underline{J} \setminus \underline{J}^\circ \\
& r_j(\tilde{\boldsymbol{\zeta}}) \leq \overline{y}_j \qquad \forall j \in \overline{J} \setminus \overline{J}^\circ \\
& \boldsymbol{r} \in \mathcal{L}(m, N_E, [N_E])
\end{array}
$$

Performance of Two-stage BDLDR

Theorem

For the two-stage problem, $Z_{BDLDR} \leq Z_{DLDR} \leq Z_{LDR}$.

- Proof is a bit tedious, but the basic idea is that the BDLDR includes the DLDR as a special case, and hence is more flexible and therefore yields a lower objective.
- Next, to generalize the BDLDR for use together with non-anticipativity requirements and expectation constraints.

BDLDR for General Model

- Define the following sets $\forall k \in [K]$:

$$\begin{array}{rcl} N^+(k) & = & \{j \in [K] : \Phi_k \subseteq \Phi_j\} \\ N^-(k) & = & \{j \in [K] : \Phi_j \subseteq \Phi_k\} \end{array}$$

- Similarly, we have two sub-problems:

$$\underline{\forall k \in [K], \forall i \in \underline{J}_k}$$

$$\begin{array}{rclll} \sum_{j \in N^+(k)} \boldsymbol{U}^j \boldsymbol{p}^{i,k,j} & = & \mathbf{0} & & \\ p_l^{i,k,j} & \geq & 0 & \forall l \in \underline{J}_j & \forall j \in N^+(k) \\ p_l^{i,k,j} & \leq & 0 & \forall l \in \overline{J}_j & \forall j \in N^+(k) \setminus \{k\} \\ p_l^{i,k,k} & \leq & 0 & \forall l \in \overline{J}_k \setminus \{i\} & \\ p_i^{i,k,k} & = & 1 & & \end{array}$$

$$\underline{\forall k \in [K], \forall i \in \overline{J}_k}$$

$$\begin{array}{rclll} \sum_{j \in N^+(k)} \boldsymbol{U}^j \boldsymbol{q}^{i,k,j} & = & \mathbf{0} & & \\ q_l^{i,k,j} & \leq & 0 & \forall l \in \overline{J}_j & \forall j \in N^+(k) \\ q_l^{i,k,j} & \geq & 0 & \forall l \in \underline{J}_j & \forall j \in N^+(k) \setminus \{k\} \\ q_l^{i,k,k} & \geq & 0 & \forall l \in \underline{J}_k \setminus \{i\} & \\ q_i^{i,k,k} & = & -1 & & \end{array}$$

Constructing the BDLDR

- Consider an LDR which satisfies the relaxed constraints

$$\begin{array}{ll} \boldsymbol{\mathcal{T}}(\tilde{\boldsymbol{\zeta}})\boldsymbol{x} + \sum_{k=1}^{K} \boldsymbol{U}^k \boldsymbol{r}^k(\tilde{\boldsymbol{\zeta}}) = \boldsymbol{\nu}(\tilde{\boldsymbol{\zeta}}) & \\ r_j^k(\tilde{\boldsymbol{\zeta}}) \geq \underline{y}_j^k & \forall k \in [K], \forall j \in \underline{J}_k \setminus \underline{J}_k^\circ \\ r_j^k(\tilde{\boldsymbol{\zeta}}) \leq \overline{y}_j^k & \forall k \in [K], \forall j \in \overline{J}_k \setminus \overline{J}_k^\circ \\ \boldsymbol{r}^k \in \mathcal{L}(m_k, N_E, \Phi_k) & \forall k \in [K] \end{array}$$

- The non-anticipative BDLDR is defined as:

$$\hat{\boldsymbol{r}}^k(\tilde{\boldsymbol{\zeta}}) \triangleq \underbrace{\boldsymbol{r}^k(\tilde{\boldsymbol{\zeta}})}_{\text{Linear Part}} + \sum_{j \in N^-(k)} \left(\sum_{i \in \underline{J}_j^\circ} \left(r_i^j(\tilde{\boldsymbol{\zeta}}) - \underline{y}_i^j \right)^- \underbrace{\boldsymbol{p}^{i,j,k}} + \sum_{i \in \overline{J}_j^\circ} \left(r_i^j(\tilde{\boldsymbol{\zeta}}) - \overline{y}_i^j \right)^+ \underbrace{\boldsymbol{q}^{i,j,k}} \right)$$

Optimal solution of i^{th} sub-problem (Deflected Part)

BDLDR Properties

Theorem

Each non-anticipative BDLDR, $\hat{\boldsymbol{r}}^k(\tilde{\boldsymbol{\zeta}})$ satisfies the following properties:

1. $\sum_{k=1}^{K} \boldsymbol{U}^k \hat{\boldsymbol{r}}^k(\tilde{\boldsymbol{\zeta}}) = \sum_{k=1}^{K} \boldsymbol{U}^k \boldsymbol{r}^k(\tilde{\boldsymbol{\zeta}})$
2. $\underline{\boldsymbol{y}}^k \leq \hat{\boldsymbol{r}}^k(\tilde{\boldsymbol{\zeta}}) \leq \overline{\boldsymbol{y}}^k, \quad \forall k \in [K]$

- For reference:

$$\hat{\boldsymbol{r}}^k(\tilde{\boldsymbol{\zeta}}) \triangleq \underbrace{\boldsymbol{r}^k(\tilde{\boldsymbol{\zeta}})}_{\text{Linear Part}} + \sum_{j \in N^-(k)} \left(\sum_{i \in \underline{J}_j^\circ} \left(r_i^j(\tilde{\boldsymbol{\zeta}}) - \underline{y}_i^j \right)^- \underbrace{\boldsymbol{p}^{i,j,k}} + \sum_{i \in \overline{J}_j^\circ} \left(r_i^j(\tilde{\boldsymbol{\zeta}}) - \overline{y}_i^j \right)^+ \underbrace{\boldsymbol{q}^{i,j,k}} \right)$$

Optimal solution of i^{th} sub-problem (Deflected Part)

Explicit BDLDR Applied to General Problem

$$
\begin{aligned}
& Z_{BDLDR} \\
= & \min_{\boldsymbol{x},\{\boldsymbol{r}^{0,k},\boldsymbol{R}^k\}_{k=1}^K} && \boldsymbol{c}^{0\prime}\boldsymbol{x} + \sum_{k=1}^K \boldsymbol{d}^{0,k\prime}\boldsymbol{r}^{0,k} + \sup_{\hat{\boldsymbol{\zeta}}\in\hat{\mathcal{V}}}\left\{\sum_{k=1}^K \boldsymbol{d}^{0,k\prime}\boldsymbol{R}^k\hat{\boldsymbol{\zeta}}\right\} \\
& && + \sum_{k=1}^K \sum_{j\in N^-(k)} \sum_{i\in \underline{J}^\circ_{0,j,k}} \pi\left(-r_i^{0,j} + \underline{y}_i^j, -\boldsymbol{R}^{j\prime}\boldsymbol{e}^i\right)\boldsymbol{d}^{0,k\prime}\boldsymbol{p}^{i,j,k} \\
& && + \sum_{k=1}^K \sum_{j\in N^-(k)} \sum_{i\in \overline{J}^\circ_{0,j,k}} \pi\left(r_i^{0,j} - \overline{y}_i^j, \boldsymbol{R}^{j\prime}\boldsymbol{e}^i\right)\boldsymbol{d}^{0,k\prime}\boldsymbol{q}^{i,j,k} \\
& \text{s.t.} && \boldsymbol{c}^{l\prime}\boldsymbol{x} + \sum_{k=1}^K \boldsymbol{d}^{l,k\prime}\boldsymbol{r}^{0,k} + \sup_{\hat{\boldsymbol{\zeta}}\in\hat{\mathcal{V}}}\left\{\sum_{k=1}^K \boldsymbol{d}^{l,k\prime}\boldsymbol{R}^k\hat{\boldsymbol{\zeta}}\right\} \\
& && + \sum_{k=1}^K \sum_{j\in N^-(k)} \sum_{i\in \underline{J}^\circ_{l,j,k}} \pi\left(-r_i^{0,j} + \underline{y}_i^j, -\boldsymbol{R}^{j\prime}\boldsymbol{e}^i\right)\boldsymbol{d}^{l,k\prime}\boldsymbol{p}^{i,j,k} \\
& && + \sum_{k=1}^K \sum_{j\in N^-(k)} \sum_{i\in \overline{J}^\circ_{l,j,k}} \pi\left(r_i^{0,j} - \overline{y}_i^j, \boldsymbol{R}^{j\prime}\boldsymbol{e}^i\right)\boldsymbol{d}^{l,k\prime}\boldsymbol{q}^{i,j,k} \leq b_l \quad \forall l\in[M]
\end{aligned}
$$

Explicit BDLDR Applied to General Problem (cont.)

$$
\begin{aligned}
\text{s.t.} \quad & \boldsymbol{\mathcal{T}}^0 \boldsymbol{x} + \sum_{k=1}^{K} \boldsymbol{U}^k \boldsymbol{r}^{0,k} = \boldsymbol{\nu}^0 \\
& \boldsymbol{\mathcal{T}}^j \boldsymbol{x} + \sum_{k=1}^{K} \boldsymbol{U}^k \boldsymbol{R}^k \boldsymbol{e}^j = \boldsymbol{\nu}^j & & \forall k \in [K], \forall j \in [N_E] \\
& r_j^{0,k} + \boldsymbol{e}^{j\prime} \boldsymbol{R}^k \boldsymbol{\zeta} \geq \underline{y}_j^k & \forall \boldsymbol{\zeta} \in \mathcal{V} & \quad \forall k \in [K], j \in \underline{J}_k \setminus \underline{J}_k^\circ \\
& r_j^{0,k} + \boldsymbol{e}^{j\prime} \boldsymbol{R}^k \boldsymbol{\zeta} \leq \overline{y}_j^k & \forall \boldsymbol{\zeta} \in \mathcal{V} & \quad \forall k \in [K], j \in \overline{J}_k \setminus \overline{J}_k^\circ \\
& \boldsymbol{x} \geq \boldsymbol{0}
\end{aligned}
$$

BDLDR Performance

Theorem

For the general problem, we have

$$Z_{GEN}{}^{*} \leq Z_{BDLDR} \leq Z_{SLDR} \leq Z_{LDR}$$

.

- The k^{th} BDLDR sums over $j \in N^{-}(k)$,
  - These are the indices which are contained by the current (k^{th}) information index set,
  - BDLDR does not violate non-anticipative of the LDR which it is based upon.

Outline

Summary

- Presented a framework for using different decision rules for Distributionally Robust Optimization.
- LDRs: Tractable approximation
  - Poor performance
- SLDRs: Improves performance
  - Retains linear structure
- BDLDRs: Improves performance beyond SLDR
  - Requires robust bounds
  - Problem becomes complex

- ROME: Robust Optimization Made Easy
- Solves problems within our DRO framework
- Comprehensive modeling examples