Distributionally Robust Optimization with ROME (part 2)

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Outline

Outline

Introduction

Basic Structure of a ROME Program

ROME Features

Example 1: Simple Test

Example 2: Inventory Management

Conclusion

What is ROME?

- ROME: Robust Optimization Made Easy
- Algebraic modeling language in the MATLAB environment for modeling Robust Optimization (RO) problems
- Primarily designed for RO problems within the DRO framework
- ROAM Design Goals
  - Environment for rapid prototyping of new RO ideas
  - Ease the transition from theory to practice
  - Ease numerical studies of RO models

What ROME is NOT

- ROME is NOT a solver engine
  - ROME calls 3rd party solvers to do actual solving (e.g. CPLEX, MOSEK, SDPT3)
  - ROME serves as an intermediary to translate an uncertain optimization program from a mathematically intuitive form into a solver-understandable form
- ROME is NOT a large-scale solver platform
  - ROME can handle medium-sized problems reasonably well, and some large problems
  - Best to write specialized code (e.g. in C, C++)

Why ROME?

- RO programs, especially with more complex decision rules (e.g. BDLDR), can be extremely complex, typically involving the following steps
  - Constructing robust counterparts
  - Finding deflected components (non-anticipative)
  - Constructing robust bounds
- Fortunately, most of these steps are mechanical in nature
  - Potential to be automated

Outline

Ingredients of a Robust Optimization Program

- Uncertainties: Methods to input distributional properties
- Variables: Deterministic variables, recourse variables
- Arithmetic operations on Uncertainties, Variables
- Conic Constraints
- Objective
- Optimization Results

The ROME Environment

- The ROME environment is started by a call to `roam_begin`
    - Sets up internal ROAM structures (invisible)
- A model in ROAM is created by a call to `roam_model`
    - Returns a handle which you can use to access model data, or call model member functions
- Input uncertainties, variables, constraints, and objective
- Call solve to complete the model and run solver.
- Can access optimization results after solve
- `roam_end` clears the ROME environment and frees memory in ROAM structures

Variables and Uncertainties

- Declared with the `newvar` keyword with different options
  - `<nothing>` : deterministic variables
  - `uncertain` : uncertainties
  - `linearrule>` : LDR recourse variables
- Can set distributional properties on uncertainty variables
- Can be construted with various sizes
  - Scalars, vectors, matrices, multi-dimensional arrays

ROME Structure Code (I)

```
% ROME_STRUCTURE.m
% File to illustrate ROME's structure. No real model to be solved.

% PREAMBLE
% Begins ROME Environment and creates model object
rome_begin;                          % Begins the ROME Environment
h = rome_model('Dummy File');        % Creates a model object

% MAIN CODE BODY
% Declare variables, objective and constraints
% Declare uncertain variables first, and set their distributional properties,
% mean, covariance, directional deviations, support. Ends with call to solve.
% Declare variables:
newvar z(5, 1) uncertain;            % Declare z as 5 x 1 uncertainty vector
z.set_mean(0);                       % Set mean of z
z.Covar = 1;                         % Set covariance of z

newvar x(3, 1);                      % Declare 3 x 1 variable x
newvar y(3, 4, z) linearrule;        % Declare 3 x 4 LDR variable y
```

Operators

- Most of MATLAB's arithmetic operators have been overloaded to work with declared ROME variables
  - Addition / Subtraction
  - Array and Matrix Multiplication
  - Subscripted Reference and Assignment (e.g. `A(1, 2)`)
  - Array shape manipulation commands (`size`, `reshape`)
- Idea: whatever you can do with MATLAB matrices, you should be able to do with ROME variables
- Non-standard operation: mean (for uncertainties and LDR variables)

Constraints and Objective

- Constraints
  - Declared using the `rome_constraint` function
  - Accepts ROME expressions containing a single inequality or equalities (examples later)
  - For LDR variables, constraints will be translated into the Robust Counterpart *automatically*
- Objective
  - One objective per model
  - Specified using `rome_minimize` or `rome_maximize`

Getting Optimization Results

- After calling `solve`,
  - Get values of variables using `eval`
  - Can get values of declared variables or even functions of variables

ROME Structure Code (II)

```
% Declare constraints:
rome_constraint(x(2:3) <= 2*x(1:2)); %
rome_constraint(sum(y, 1) == 1);     % Various constraints can be applied.
rome_constraint(y(:, 1) + x >= 2);   % ROME variables can be manipulated by
rome_constraint(norm2(x) <= 1);      % standard arithmetic operations.
rome_constraint(mean(y(2, :)) <= 5); %

% Declare objective:
rome_minimize(sum(x) + sum(y(:, 2))); % objective

% Instruct ROME to solve
h.solve;                             % Terminate the current model with solve

% GET RESULTS
% Get optimization results
y_sol = h.eval(y);                   % Get y
u_sol = h.eval(y(2, 2) + x(1));      % Get function of variables

% CLEANUP
% Clear up ROME memory
rome_end;                            % Complete modeling and deallocate ROME memory
```

Outline

Why Use ROME?

- Uncertainty description
- LDRs as primitives variables in ROAM
- Non-anticipative requirements
- In-built support for BDLDRs
- Numerical analysis of results

Uncertainty Description

- Can set distributional properties of uncertainties
- Stored in ROAM's memory and invoked when necessary
  - e.g. Robust Counterpart for inequalities on LDRs
  - e.g. Robust Bounds

```
newvar z(10, 1)   uncertain;  % create 10 x 1 uncertainty vector

rome_constraint(z.mean == 1); % Sets the mean to be exactly 1

rome_constraint(z >= 0);      % Sets the support of z to be
rome_constraint(z <= 2);      % between 0 and 2 component-wise

z.Covar = eye(10);            % Sets an identity covariance matrix

z.FDev = 1.8*ones(10, 1);     % Sets the forward deviation
z.BDev = 1.8*ones(10, 1);     % Sets the backward deviation
```

LDRs as Primitive Objects

- LDRs are declared and manipulated directly
  - Much more mathematically meaningful
  - Don't have to worry about internal structure of LDR or worry about how to write the robust counterpart
  - Works in concert with the uncertainty description

```
newvar z(5)          uncertain ;  % make uncertainty vector
newvar y1(z)         linearrule;  % create scalar LDR
newvar y2(3, z)      linearrule;  % create 3 x 1 LDR
newvar y3(4, 3, z)   linearrule;  % create 4 x 3 LDR
```

Non-anticipative Requirements

- In ROME, you can make variables with a prescribed dependency pattern on the uncertainties

```
newvar z(N) uncertain;                        % create an uncertainty vector
y = [];                                       % allocate an empty matrix
for ii = 1:N
    newvar tmp(1, z(1:ii)) linearrule;        % create an LDR in each iteration
    y = [y; tmp];                             % append to the output matrix
end
```

- e.g. $\tilde{z}$ where the i^{th} component depends on the first i components of uncertainty
- More directly, can specify dependency pattern at creation

```
pY = [true(N, 1), tril(true(N, N))];          % augmented lower triangular pattern
newvar z(N) uncertain;                        % create an uncertainty vector
newvar y(N, z, 'Pattern', pY) linearrule;     % construct the LDR with the pattern
```

In-built-support for BDLDRs

- As we saw before, BDLDRs improved over LDRs, but were terribly complicated, especially for the non-anticipative case. Recall the steps involved:
  1. Form and solve the two sub-problems
  2. Remove “unnecessary” inequality constraints
  3. Make the BDLDR and construct robust bounds
  4. Solve the final problem
- Design Concept: while the BDLDR is complex, it’s just another decision rule, you shouldn’t have to change your model to use the BDLDR
- Just call `solve_deflected`, instead of `solve`

Numerical Analysis of Results (I)

- For deterministic optimization software packages this is trivial
- Recall that recourse decisions are *functions* of uncertainties
- Most general setting in ROME: deflected variable

$$\hat{\boldsymbol{x}}(\tilde{\boldsymbol{z}}) = \left(\boldsymbol{x}^0 + \boldsymbol{X}\tilde{\boldsymbol{z}}\right) + \boldsymbol{P}\left(\boldsymbol{y}^0 + \boldsymbol{Y}\tilde{\boldsymbol{z}}\right)^-$$

- `eval` function returns an object, which encodes these parameters.
- Use `linearpart` and `deflectedpart` functions

Numerical Analysis of Results (II)

- ROME also includes a "prettyprint" functionality to aid debugging and prototyping of small problems, e.g.

```
y_sol =

-0.000 + 1.000*z1 + 1.00(-0.000 + 1.000*z1)^- - 1.00(1.000 - 1.000*z1)^-
```

- Instantiate solution of uncertainty values using `insert`
- Use as a prescriptive tool or for Monte-Carlo simulation

```
x_sol = h.eval(x);          % Get the solution object
z_vals = randn(N, 100);     % Draw r.v.s from standard normal distribution
x_vals = zeros(M, 100);     % Allocate output matrix
for ii = 1:100
    % instantiate solution with r.v.
    x_vals(:, ii) = x_sol.insert(z_vals(:, ii));
end
```

Outline

Recall Motivating Problem for BDLDR

$$
\begin{array}{rl}
\min\limits_{y(\cdot)} & \sup\limits_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(|y(\tilde{z}) - \tilde{z}|\right) \\
& 0 \leq y(\tilde{z}) \leq 1 \\
& y \in \mathcal{Y}\left(1, 1, \{1\}\right) \\
& \Updownarrow \\
\min\limits_{y(\cdot),u(\cdot),v(\cdot)} & \sup\limits_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(u(\tilde{z}) + v(\tilde{z})\right) \\
\text{s.t.} & u(\tilde{z}) - v(\tilde{z}) = y(\tilde{z}) - \tilde{z} \\
& 0 \leq y(\tilde{z}) \leq 1 \\
& u(\tilde{z}), v(\tilde{z}) \geq 0 \\
& y, u, v \in \mathcal{Y}\left(1, 1, \{1\}\right)
\end{array}
$$

- Where we have used the identities $x = x^+ - x^-, |x| = x^+ + x^-$.

ROME Code for Simple Example (I)

```
% SIMPLE_EXAMPLE.m
% Script to demonstrate several key features and functions in ROME,
% and illustrate overall structure of a ROME program

% ROME MODELING CODE
% Preamble
rome_begin;                          % Begins the ROME Environment
h = rome_model('Simple Example');    % Creates a model object

% Feature 1: Handling Uncertainties
newvar z uncertain;                  % Declare z as a scalar uncertainty
z.set_mean(0);                       % Set distributional properties of z
z.Covar = 1;                         % zero mean and unit variance.

% Feature 2: Declaring LDRs as primitive objects
newvar u(z) v(z) linearrule nonneg;  % nonnegative LDRs u and v
newvar y(z) linearrule;              % LDR y
```

$$
\begin{array}{rl}
\min\limits_{y(\cdot),u(\cdot),v(\cdot)} & \sup\limits_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(u(\tilde{z}) + v(\tilde{z})\right) \\
\text{s.t.} & u(\tilde{z}) - v(\tilde{z}) = y(\tilde{z}) - \tilde{z} \\
& 0 \leq y(\tilde{z}) \leq 1 \\
& u(\tilde{z}), v(\tilde{z}) \geq 0 \\
& y, u, v \in \mathcal{Y}\left(1, 1, \{1\}\right)
\end{array}
$$

ROME Code for Simple Example (II)

```
% Objective
rome_minimize(mean(u + v));

% Constraints
rome_constraint(u - v == y - z);        % equality constraint
rome_box(y, 0, 1);                      % 0 <= y <= 1 constraint

% Feature 4: Support for BDLDRs
h.solve_deflected;                 % Uses BDLDR as decision rules
% h.solve;                         % Use this instead to use basic LDRs as decision rules

% Feature 5: Extract solution for analysis
y_sol = h.eval(y);                 % Call 'eval' after solve and before rome_end

% Complete modeling and deallocate ROME memory
rome_end;
```

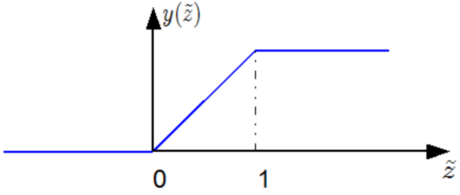
$$
\begin{array}{rl}
\min\limits_{y(\cdot),u(\cdot),v(\cdot)} & \sup\limits_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(u(\tilde{z}) + v(\tilde{z})\right) \\
\text{s.t.} & u(\tilde{z}) - v(\tilde{z}) = y(\tilde{z}) - \tilde{z} \\
& 0 \leq y(\tilde{z}) \leq 1 \\
& u(\tilde{z}), v(\tilde{z}) \geq 0 \\
& y, u, v \in \mathcal{Y}\left(1, 1, \{1\}\right)
\end{array}
$$

Solution

```
y_sol =

-0.000 + 1.000*z1 + 1.00(-0.000 + 1.000*z1)^- - 1.00(1.000 - 1.000*z1)^-
```

$$
\begin{aligned}
\hat{y}_{sol}(\tilde{z}) &= \tilde{z} + \tilde{z}^{-} - (1 - \tilde{z})^{+} \\
&= \tilde{z}^{+} - (1 - \tilde{z})^{+} \\
&= \begin{cases} \tilde{z} & \text{if } 0 \leq \tilde{z} \leq 1 \\ 0 & \text{if } \tilde{z} \leq 0 \\ 1 & \text{if } \tilde{z} \geq 1 \end{cases}
\end{aligned}
$$



Outline

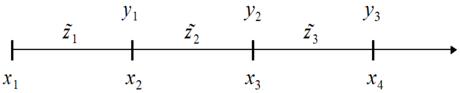
Robust Inventory Model

- Model a distribution-free, multi-period, inventory control problem with service constraints.
- Demand is exogenous, with unknown, but partially characterized distribution in a family $\mathbb{F}$, defined by:
  - Covariance Matrix: Temporal demand correlation can be modeled by a non-diagonal covariance matrix.
  - Mean: Assume fixed mean (for simplicity).
  - Support: Maximum demand in each period.
- Backorders allowed, but in some applications, a penalty cost might not be a good model for stockouts.
  - In our model, we avoid stockouts with a constraint on the fill-rate.
  - Fill-rate constraint acts as a service guarantee to the consumers.

Model Parameters

Parameters

Num Periods	:	$T \in \mathbb{N}$
Order Cost	:	$\boldsymbol{c} \in \Re^T$
Holding Cost	:	$\boldsymbol{h} \in \Re^T$
Min Fill Rate	:	$\boldsymbol{\beta} \in \Re^T$
Max Order Qty	:	$\boldsymbol{x}^{MAX} \in \Re^T$



Uncertainties

Demand	:	$\tilde{\boldsymbol{z}} \in \Re^T$

Decisions

Order Quantity	:	$\boldsymbol{x}(\tilde{\boldsymbol{z}}) : x_t(\tilde{\boldsymbol{z}}) \in \mathcal{L}(1, T, [t-1])$
Inventory Level	:	$\boldsymbol{y}(\tilde{\boldsymbol{z}}) : y_t(\tilde{\boldsymbol{z}}) \in \mathcal{L}(1, T, [t])$

Robust Fill Rate Constraint

$$\text{Fill Rate} = \frac{\text{Expected Sales}}{\text{Expected Demand}} \geq \beta$$

- Using our notation, the robust (worst-case) version:

$$\inf_{\mathbb{P} \in \mathbb{F}} \mathrm{E}_{\mathbb{P}} \left(\min \left\{ \tilde{z}_t, y_{t-1}(\tilde{\boldsymbol{z}}) + x_t(\tilde{\boldsymbol{z}}) \right\} \right) \geq \beta_t \mu_t$$

- Apply inventory balance equation and re-arrange:

$$\sup_{\mathbb{P} \in \mathbb{F}} \mathrm{E}_{\mathbb{P}} \left(y_t(\tilde{\boldsymbol{z}})^- \right) \leq (1 - \beta_t) \mu_t$$

Model Formulation

- Family of uncertainties:

$$\mathbb{F} = \left\{ \mathbb{P} : \mathrm{E}_{\mathbb{P}}\left(\tilde{\boldsymbol{z}}\right) = \boldsymbol{\mu}, \mathrm{E}_{\mathbb{P}}\left(\tilde{\boldsymbol{z}}\tilde{\boldsymbol{z}}'\right) = \boldsymbol{\Sigma} + \boldsymbol{\mu}\boldsymbol{\mu}', \mathbb{P}\left(0 \leq \tilde{\boldsymbol{z}} \leq \boldsymbol{z}^{MAX}\right) = 1 \right\}$$

- Robust Inventory Model with Fill Rate constraints:

$$\begin{array}{ll} \displaystyle\min_{\boldsymbol{x}(\cdot),\boldsymbol{y}(\cdot)} & \displaystyle\sup_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\boldsymbol{c}'\boldsymbol{x}(\tilde{\boldsymbol{z}}) + \boldsymbol{h}'\left(\boldsymbol{y}(\tilde{\boldsymbol{z}})\right)^{+}\right) \\ \text{s.t.} & \displaystyle\sup_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\boldsymbol{y}(\tilde{\boldsymbol{z}})^{-}\right) \leq \left(\boldsymbol{I} - \textbf{diag}\left(\boldsymbol{\beta}\right)\right)\boldsymbol{\mu} \\ & \boldsymbol{D}\boldsymbol{y}\left(\tilde{\boldsymbol{z}}\right) - \boldsymbol{x}\left(\tilde{\boldsymbol{z}}\right) = -\tilde{\boldsymbol{z}} \\ & \boldsymbol{0} \leq \boldsymbol{x}\left(\tilde{\boldsymbol{z}}\right) \leq \boldsymbol{x}^{MAX} \\ & x_t \in \mathcal{L}(1, T, [t-1]) \quad \forall t \in [T] \\ & y_t \in \mathcal{L}(1, T, [t]) \quad \forall t \in [T] \end{array}$$

$$\boldsymbol{D} = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ -1 & 1 & 0 & \ldots & 0 \\ 0 & -1 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}$$

Model Transformed into DRO Framework

- After linearizing piecewise-linear terms,

$$
\begin{array}{lll}
\min\limits_{\boldsymbol{x}(\cdot),\boldsymbol{y}(\cdot),\boldsymbol{r}(\cdot),\boldsymbol{s}(\cdot)} & \sup\limits_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}} \left(\boldsymbol{c}'\boldsymbol{x}(\tilde{\boldsymbol{z}}) + \boldsymbol{h}'\boldsymbol{r}(\tilde{\boldsymbol{z}})\right) & \\
\text{s.t.} & \sup\limits_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}} \left(\boldsymbol{s}(\tilde{\boldsymbol{z}})\right) \leq (\boldsymbol{I} - \textbf{diag}\,(\boldsymbol{\beta}))\boldsymbol{\mu} & \\
& \boldsymbol{r}\left(\tilde{\boldsymbol{z}}\right) \geq \boldsymbol{y}\left(\tilde{\boldsymbol{z}}\right) & \\
& \boldsymbol{s}\left(\tilde{\boldsymbol{z}}\right) \geq -\boldsymbol{y}\left(\tilde{\boldsymbol{z}}\right) & \\
& \boldsymbol{r}\left(\tilde{\boldsymbol{z}}\right), \boldsymbol{s}\left(\tilde{\boldsymbol{z}}\right) \geq \boldsymbol{0} & \\
& \boldsymbol{D}\boldsymbol{y}\left(\tilde{\boldsymbol{z}}\right) - \boldsymbol{x}\left(\tilde{\boldsymbol{z}}\right) = -\tilde{\boldsymbol{z}} & \\
& \boldsymbol{0} \leq \boldsymbol{x}\left(\tilde{\boldsymbol{z}}\right) \leq \boldsymbol{x}^{MAX} & \\
& x_t \in \mathcal{L}(1, T, [t-1]) & \forall t \in [T] \\
& r_t, s_t, y_t \in \mathcal{L}(1, T, [t]) & \forall t \in [T]
\end{array}
$$

ROME Code for Inventory Problem

```
% model parameters
T = 10;                    % planning horizon
c = 1 *ones(T, 1);         % order cost rate
hcost = 2*ones(T, 1);      % holding cost rate
beta  = 0.50*ones(T, 1); % minimum fillrate in each period
xMax  = 100*ones(T, 1);  % maximum order quantity in each period
alpha = 0.5;               % temporal autocorrelation factor
L = alpha * tril(ones(T), -1) + eye(T); % autocorrelation matrix

% numerical uncertainty parameters
zMax  = 105*ones(T, 1);    % maximum demand in each period
zMean = 30*ones(T, 1);     % mean demand in each period
zCovar = 20*(L * L');      % temporal demand covariance

% differencing matrix
D = eye(T) - diag(ones(T-1 , 1), -1);

% dependency structure
pX = logical([tril(ones(T)), zeros(T, 1)]);
```

ROME Code for Inventory Problem (cont.)

```
% Step 3: BDLDR Method
% --------------------
h = rome_begin('Robust Inventory (BDLDR)');
tic;

% declare uncertainties
newvar z(T) uncertain nonneg;

% define uncertainty parameters
rome_constraint(z <= zMax);   % support
z.set_mean(zMean);            % mean
z.Covar = zCovar;             % covariance

% define LDR variables
newvar x(T, z, 'Pattern', pX) linearrule;  % order quantity
newvar y(T, z) linearrule;                 % inventory level

% define auxilliary variables
newvar r(T, z) s(T, z) linearrule nonneg;
```

$$\mathbb{F} = \left\{ \mathbb{P} : \begin{array}{l} \mathbb{P}\left(0 \le \tilde{\boldsymbol{z}} \le \boldsymbol{z}^{MAX}\right) = 1 \\ \mathrm{E}_{\mathbb{P}}\left(\tilde{\boldsymbol{z}}\right) = \boldsymbol{\mu}, \\ \mathrm{E}_{\mathbb{P}}\left(\tilde{\boldsymbol{z}}\tilde{\boldsymbol{z}}'\right) = \boldsymbol{\Sigma} + \boldsymbol{\mu}\boldsymbol{\mu}' \end{array} \right\}$$

$$\begin{array}{ll} \boldsymbol{r}\left(\tilde{\boldsymbol{z}}\right), \boldsymbol{s}\left(\tilde{\boldsymbol{z}}\right) \ge \mathbf{0} & \\ x_t \in \mathcal{L}(1, T, [t-1]) & \forall t \in [T] \\ r_t, s_t, y_t \in \mathcal{L}(1, T, [t]) & \forall t \in [T] \end{array}$$

ROME Code for Inventory Problem (cont.)

```
% auxilliary constraints
rome_constraint(r >=  y); % since r >= y^+
rome_constraint(s >= -y); % since s >= y^-

% fillrate constraint
rome_constraint(mean(s) <= diag(ones(T, 1) - beta) * zMean);

% inventory balance constraint
rome_constraint(D*y == x - z);

% order quantity constraints
rome_box(x, 0, xMax);

% objective
rome_minimize(c'*mean(x) + hcost'*mean(r));

% solve and display optimal objective
h.solve_deflected;
disp(sprintf('BDLDR Obj = %0.2f, time = %0.2f secs', h.ObjVal, toc));
x_sol_bdldr = h.eval(x)

rome_end;
```

$$
\begin{array}{cl}
\min\limits_{\boldsymbol{x}(\cdot),\boldsymbol{y}(\cdot),\boldsymbol{r}(\cdot),\boldsymbol{s}(\cdot)} & \sup\limits_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\boldsymbol{c}'\boldsymbol{x}(\tilde{\boldsymbol{z}}) + \boldsymbol{h}'\boldsymbol{r}(\tilde{\boldsymbol{z}})\right) \\
\text{s.t.} & \sup\limits_{\mathbb{P}\in\mathbb{F}} \mathrm{E}_{\mathbb{P}}\left(\boldsymbol{s}(\tilde{\boldsymbol{z}})\right) \leq \left(\boldsymbol{I} - \mathbf{diag}\left(\boldsymbol{\beta}\right)\right)\boldsymbol{\mu} \\
& \boldsymbol{D}\boldsymbol{y}\left(\tilde{\boldsymbol{z}}\right) - \boldsymbol{x}\left(\tilde{\boldsymbol{z}}\right) = -\tilde{\boldsymbol{z}} \\
& \boldsymbol{r}\left(\tilde{\boldsymbol{z}}\right) \geq \boldsymbol{y}\left(\tilde{\boldsymbol{z}}\right) \\
& \boldsymbol{s}\left(\tilde{\boldsymbol{z}}\right) \geq -\boldsymbol{y}\left(\tilde{\boldsymbol{z}}\right) \\
& \mathbf{0} \leq \boldsymbol{x}\left(\tilde{\boldsymbol{z}}\right) \leq \boldsymbol{x}^{MAX}
\end{array}
$$

Numerical Example (I)

- Use a covariance matrix which represents temporal autocorrelation of demand, $\boldsymbol{\Sigma} = \sigma \boldsymbol{L}(\alpha)\boldsymbol{L}(\alpha)'$

$$\boldsymbol{L}(\alpha) = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \alpha & 1 & 0 & \dots & 0 \\ \alpha & \alpha & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha & \alpha & \alpha & \dots & 1 \end{bmatrix}$$

- $(z^{MAX}, \beta) = (100, 0.5)$

h	μ	α	$Z_{LDR}{}^{(2)}$	Z_{LDR}	Z_{BDLDR}	h	μ	α	$Z_{LDR}{}^{(2)}$	Z_{LDR}	Z_{BDLDR}
1	10	0.00	590.0	106.0	106.0	3	10	0.00	1490.0	126.0	126.0
1	10	0.20	590.0	108.2	108.1	3	10	0.20	1490.0	131.8	131.5
1	10	0.50	590.0	119.9	117.2	3	10	0.50	1490.0	163.0	155.9
1	10	0.90	590.0	153.3	135.8	3	10	0.90	1490.0	251.8	205.4

Numerical Example (II)

- $(z^{MAX}, \beta) = (200, 0.5)$

h	μ	α	$Z_{LDR}^{(2)}$	Z_{LDR}	Z_{BDLDR}	h	μ	α	$Z_{LDR}^{(2)}$	Z_{LDR}	Z_{BDLDR}
1	10	0.00	$+\infty$	121.0	107.0	3	10	0.00	$+\infty$	166.1	128.0
1	10	0.20	$+\infty$	167.9	109.3	3	10	0.20	$+\infty$	285.9	133.8
1	10	0.50	$+\infty$	313.6	119.4	3	10	0.50	$+\infty$	654.1	159.9
1	10	0.90	$+\infty$	722.3	141.6	3	10	0.90	$+\infty$	1668.7	216.7

- $(z^{MAX}, \beta) = (200, 0.8)$

h	μ	α	$Z_{LDR}^{(2)}$	Z_{LDR}	Z_{BDLDR}	h	μ	α	$Z_{LDR}^{(2)}$	Z_{LDR}	Z_{BDLDR}
1	10	0.00	$+\infty$	163.4	129.5	3	10	0.00	$+\infty$	276.6	185.8
1	10	0.20	$+\infty$	285.3	136.0	3	10	0.20	$+\infty$	588.2	202.4
1	10	0.50	$+\infty$	722.1	161.5	3	10	0.50	$+\infty$	1681.6	267.8
1	10	0.90	$+\infty$	3402.0	2563.5	3	10	0.90	$+\infty$	9192.7	7019.9

ROME Output (MOSEK Solver)

```
EDU>> inventory_fillrate_example
Status: OPTIMAL
LDR Obj = 1747.50, time = 0.25 secs

x_sol_ldr =

 97.500
 0.000 + 0.952*z1
-0.000 + 0.007*z1 + 0.945*z2
 0.000 + 0.005*z1 + 0.010*z2 + 0.937*z3
-0.000 + 0.004*z1 + 0.006*z2 + 0.013*z3 + 0.929*z4
 0.000 + 0.003*z1 + 0.005*z2 + 0.008*z3 + 0.015*z4 + 0.921*z5
-0.000 + 0.004*z1 + 0.004*z2 + 0.006*z3 + 0.009*z4 + 0.018*z5 + 0.911*z6
-0.000 + 0.003*z1 + 0.005*z2 + 0.006*z3 + 0.008*z4 + 0.012*z5 + 0.023*z6 + 0.897*z7
 0.000 + 0.004*z1 + 0.005*z2 + 0.006*z3 + 0.009*z4 + 0.011*z5 + 0.016*z6 + 0.031*z7 + 0.870*z8
-0.000 + 0.007*z1 + 0.007*z2 + 0.009*z3 + 0.011*z4 + 0.015*z5 + 0.019*z6 + 0.031*z7 + 0.058*z8 + 0.795*z9

Status: NEAR_OPTIMAL
BDLDR Obj = 300.48, time = 0.58 secs

x_sol_bdldr =

 15.333
-0.000 + 1.003*z1 + 1.00(-0.000 + 1.003*z1)^- - 1.00(100.000 - 1.003*z1)^-
-0.000 + 0.000*z1 + 1.004*z2 + 1.00(-0.000 + 0.000*z1 + 1.004*z2)^- - 1.00(100.000 - 0.000*z1 - 1.004*z2)^-
-0.000 + 0.000*z1 - 0.000*z2 + 1.004*z3 + 1.00(-0.000 + 0.000*z1 - 0.000*z2 + 1.004*z3)^- - 1.00(100.000 - 0.00
-0.000 - 0.000*z1 - 0.000*z2 + 0.000*z3 + 1.004*z4 + 1.00(-0.000 - 0.000*z1 - 0.000*z2 + 0.000*z3 + 1.004*z4)^-
 0.000 - 0.000*z1 - 0.000*z2 - 0.000*z3 + 0.000*z4 + 1.004*z5 + 1.00(0.000 - 0.000*z1 - 0.000*z2 - 0.000*z3 + (
-0.000 + 0.000*z1 - 0.000*z2 + 0.000*z3 + 0.000*z4 - 0.000*z5 + 1.004*z6 + 1.00(-0.000 + 0.000*z1 - 0.000*z2 +
 0.000 - 0.000*z1 - 0.000*z2 + 0.000*z3 + 0.000*z4 - 0.000*z5 - 0.000*z6 + 1.004*z7 + 1.00(0.000 - 0.000*z1 - (
-0.000 + 0.000*z1 - 0.000*z2 + 0.000*z3 + 0.000*z4 + 0.000*z5 - 0.000*z6 - 0.000*z7 + 1.004*z8 + 1.00(-0.000 +
-0.000 + 0.000*z1 + 0.000*z2 - 0.000*z3 + 0.000*z4 + 0.000*z5 + 0.000*z6 + 0.000*z7 + 0.000*z8 + 1.005*z9 + 1.(
```

Outline

Summary

- We have seen how ROME can be used to model Distributionally Robust Optimization problems
- ROME contains some components common to most modeling languages, and unique features for Robust Optimization
- Service-constrained inventory control example